Cosmic-Ray Problem Solutions.

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1. In this case, the transport equation becomes

$$\frac{\partial f}{\partial t} = 0 = \frac{\kappa_0}{r^2}\frac{\partial}{\partial r}r^3\frac{\partial f}{\partial r} - V_w\frac{\partial f}{\partial r} + \frac{2V_w}{3r}p\frac{\partial f}{\partial p}$$

Clearly, the momentum spectrum remains the same power law in momentum with index γ.

The equation is homogeneous in radius r, so the solution is a power law in radius. Setting $f = Ar^{\alpha}p^{-\gamma}$, the equation becomes

$$\kappa_0\left[\alpha(\alpha+2)\right] - V_w\alpha - \frac{2V_w\gamma}{3} = 0$$

or

$$\alpha^2 + \left(2 - \frac{2V_w}{\kappa_0}\right)\alpha - \frac{2V_w\gamma}{3\kappa_0} = 0$$

This is a quadratic equation for α. The general solution is

$$\alpha = \frac{V_w}{\kappa_0} - 1 \pm \sqrt{\left(\frac{V_w}{\kappa_0} - 1\right)^2 + \frac{2V_w\gamma}{3\kappa_0}}.$$

Clearly, we must choose the minus sign in front of the square root, which then yields the solution

$$f(r,p) = Ar^{-\alpha}p^{-\gamma},$$

with

$$\alpha = \frac{V_w}{\kappa_0} - 1 - \sqrt{\left(\frac{V_w}{\kappa_0} - 1\right)^2 + \frac{2V_w\gamma}{3\kappa_0}}.$$

2. Write the Parker spiral magnetic field as:

$$\mathbf{B} = B_0\left(\frac{r_0}{r}\right)^2\left[\mathbf{e}_r - \frac{r\Omega_{\odot}sin(\theta)}{V_w}\mathbf{e}_\phi\right]\left[1 - H(\theta - \frac{\pi}{2})\right],$$

where H is the Heaviside step function, and we define $\Gamma = r\Omega_{\odot}sin(\theta)/V_w)$. The drift velocity, averaged over a nearly isotropic angular distribution at a given momentum p is:

$$V_d = \frac{pcw}{3q}\nabla\times\left[\frac{\mathbf{B}}{B^2}\right].$$

The curl of a vector $\mathbf{A}$ in spherical coordinates is

$$\begin{aligned}\nabla \times \mathbf{A} &= \frac{1}{rsin(\theta)}\left[\frac{\partial}{\partial\theta}(sin(\theta)A_\phi) - \frac{\partial}{\partial\phi}A_\theta\right]\mathbf{e}_r + \left[\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}A_r - \frac{1}{r}\frac{\partial}{\partial r}rA_\phi\right]\mathbf{e}_\theta \\ &+ \left[\frac{1}{r}\frac{\partial}{\partial r}rA_\phi - \frac{1}{r}\frac{\partial}{\partial\theta}A_r\right]\mathbf{e}_\phi.\end{aligned}$$

from which we readily obtain the answer

$$\begin{aligned}V_d &= \frac{2wpcr}{3B_0 r_c^2 q(1+\Gamma^2)^2}\left[-\frac{\Gamma}{tan(\theta)}\mathbf{e}_r + (2+\Gamma^2)\mathbf{e}_\theta + \frac{\Gamma^2}{tan(\theta)}\mathbf{e}_\phi\right]\left[1 - H(\theta - \frac{\pi}{2})\right] \\ &+\delta(\theta - \pi/2)\frac{2wpcr}{3B_0 r_c^2 q}\frac{\Omega_\odot r^2}{1+\Gamma^2}\left[\mathbf{e}_r - \frac{V_w}{r\Omega_\odot}\right]\end{aligned}$$

3. Again we begin with the Parker transport equation. For a steady, spherically symmetric heliosphere with a boundary at radius r_1, neglecting the termination shock, we must solve

$$\frac{\partial f}{\partial t} = 0 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}r^2\kappa_{rr}\frac{\partial f}{\partial r}\right] - V_w\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Vw)\frac{p}{3r}\frac{\partial f}{\partial p}$$

subject to f $\rightarrow Ap^{-\gamma}$ as $r \rightarrow r_1$. To lowest order in the parameter η, we may write

$$f(r,p) = f_\infty(p)\left[1 + f_1(r,p) + f_2(r,p)... \quad \right]$$

where $f_1(r,p)$ is of first order in η, etc.

Putting this into the transport equation and keeping only the first-order terms, we have

$$\frac{f_\infty(p)}{r^2}\left[\frac{\partial}{\partial r}r^2\kappa_{rr}\frac{\partial f_1(r,p)}{\partial r}\right] = -\frac{1}{3r^2}\left(\frac{\partial r^2 V_w}{\partial r}\right)p\frac{\partial f_\infty}{\partial p}.$$

This may be manipulated to yield

$$\frac{\partial}{\partial r}r^2\kappa_{rr}\frac{\partial f_1(r,p)}{\partial r} = -\frac{1}{f_\infty(p)}\left(\frac{\partial r^2 V_w}{\partial r}\right)\frac{p}{3}\frac{\partial f_\infty}{\partial p}$$

which, in turn, yields

$$\frac{\partial f_1(r,p)}{\partial r} = -\frac{Vw}{3\kappa_{rr}}\frac{\partial ln(f_\infty)}{\partial ln(p)},$$

or, finally,

$$f(r,p) = f_\infty(p)\left[1 - \frac{1}{3}\frac{\partial ln(f_\infty)}{\partial ln(p)}\int_r^{r_1}\frac{V_w(r')}{\kappa_{rr}(r',p')}dr'\right].$$

4. We consider the one-dimensional version of Parker's transport equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left[\kappa\frac{\partial f}{\partial x}\right] - V\frac{\partial f}{\partial x} + \frac{1}{3}\frac{\partial V}{\partial x}p\frac{\partial f}{\partial p}.$$

working in the frame at rest with respect to the shock, the velocity as a function of x may be written $V(x) = [(V_1 + V_2) - (V1 + V2)H(x)]$, where $H(x)$ is the Heaviside step function, where V_1, V_2 are the upstream and downstream flow speeds, respectively. The solutions for f upstream and downstream are then

$$\begin{aligned} f_1(x,p) &= A_1(p) + B_1(p)e^{V_1 x/\kappa} \quad x < 0 \\ f_2(x,p) &= A_2(p) + B_2(p)e^{V_2 x/\kappa} \quad x > 0 \end{aligned}$$

To avoid ∞ at large positive x, we must set $B_2(p) = 0$. To fit the far upstream boundary condition, we set $A_1(p) = \alpha p^{-\gamma}$. To match the solutions at x = 0, we set $A_1(p) + B_1(p) = A_2(p)$.

Finally, we integrate the transport equation from just downstream to just upstream $(-\epsilon\ to + \epsilon)$ to obtain

$$-B_1(p)V_1 + \frac{(V_2 - V_1)}{3} p \frac{\partial A_2}{\partial p} = 0.$$

So, we may eliminate B_1 and A_1 to obtain an equation for $A_2(p)$. If we set $\eta = ln(p)$,

$$\frac{\partial A_2(p)}{\partial \eta} + \frac{3V_1}{V_1 - V_2} A_2(p) = -\alpha \frac{3V_1}{V_1 - V_2} e^{-\gamma\eta}$$

which may be rewritten as

$$\frac{\partial}{\partial \eta} e^{\frac{3V_1}{V_1 - V_2}\eta} A_2(p) = e^{(-\gamma + \frac{3V_1}{V_1 - V_2})\eta}.$$

Since, for any constant a $e^a \eta = p^a$, if we define $q = 3V_1/(V_1 - V_2)$, this may be readily integrated to yield the solution

$$A_2(p) = f(x = 0, p) = a_1 p^{-q} + \frac{\alpha q}{\alpha + q} p^{-\gamma}$$

If there is no source of low-energy particles at the shock $a_1 = 0$. If we had particles injected and accelerated at the shock, a_1 would be non-zero.

5. We proceed as in problem 4. But now there is an antisymmetric term to the diffusion tensor $\epsilon_{ijk}\kappa_A B_k/B^2$ (note, this is in standard index notation and ϵ_{ijk} is the totally asymmetric Levi-Civita epsilon - interchanging any of the indices changes the sign). The divergence of this term gives the standard drift velocity

$$\mathbf{V_d} = \frac{pcw}{3q} \nabla \times \frac{\mathbf{B}}{B^2}$$

Evaluating this at the shock yields a delta function singularity.

$$\mathbf{V}_d = \mathbf{e}_y \frac{pcw}{3q} \left(\frac{B_{z1}}{B_1^2} - \frac{B_{z2}}{B_2^2} \right) \delta(x - x_{shock}$$

The general solutions upstream and downstream are more complicated because of the y-dependence. However, we may obtain a jump condition which contains the effect of the discontinuity at the shock again, by integrating from just downstream to just upstream. Since the magnetic field **B** also changes, we obtain the jump condition

$$\left[\kappa_{xx}\frac{\partial f}{\partial x} + \frac{V_x}{3}p\frac{\partial f}{\partial p} - \frac{pcw}{3q}\frac{B_x}{B^2}\frac{\partial f}{\partial y}\right]_1^2$$

where the bracketed terms downstream of the shock are to be subtracted from the ones upstream. The term involving the change in magnetic field contains the effect of drifts at the shock.

6. We proceed as in the linear case, but for spherical geometry and consider the regions upstream and downstream of the shock differently. They are later connected by a jump condition. We set $\kappa_{rr} = \kappa_0 r$ (note that κ_0 has the dimensions of a velocity).

   Upstream, we have the same equation as in problem (1).

$$\frac{\partial f}{\partial t} = 0 = \kappa_0\frac{\partial}{\partial r}\left[r^3\frac{\partial f}{\partial r}\right] - V_w\frac{\partial f}{\partial r} + \frac{2V_w}{3r}p\frac{\partial f}{\partial t}$$

   The solution is again proportional to $r^\alpha p^\gamma$, where the value of α is different for the two regions, inside and outside of the termination shock. Upon substituting $r^\alpha p^\gamma$ into the transport equation, we find that (defining $\eta = V_w/\kappa_0$,

$$\begin{aligned} \alpha_1 &= \eta \qquad\qquad\qquad r > R_{sh} \\ \alpha_2 &= \frac{\eta}{2} - 1 \pm \sqrt{\frac{\eta}{2}^2 - \frac{2}{3}\eta\gamma} \quad r < R_{sh}. \end{aligned} \tag{1}$$

   Now consider the two parts separately.

   a. Since the only effect of the termination shock is to change the speed of the outward flow and the value of α, we must simply match the two solutions at the shock. We obtain

$$\begin{aligned} f &= A\left(\frac{r}{R_b}\right)^\eta \qquad r > R_{sh} \\ f &= A\left(\frac{R_{sh}}{R_b}\right)^\eta\left(\frac{r}{R_{sh}}\right)^{\alpha_2} \quad R < R_{sh}. \end{aligned} \tag{2}$$

   Now consider part 2. of the problem where particles are accelerated at the termination shock.

   The solution is again given in terms of power laws in radius, and the power laws are again α_1 and α_2. In addition, a constant is a solution. With the solution going to zero at small radii and at the boundary R_b, and matching the solution at the shock, we may write

$$f = A\left[1 - \left(\frac{r}{R_b}\right)^\eta\right] \qquad r > R_{sh}$$

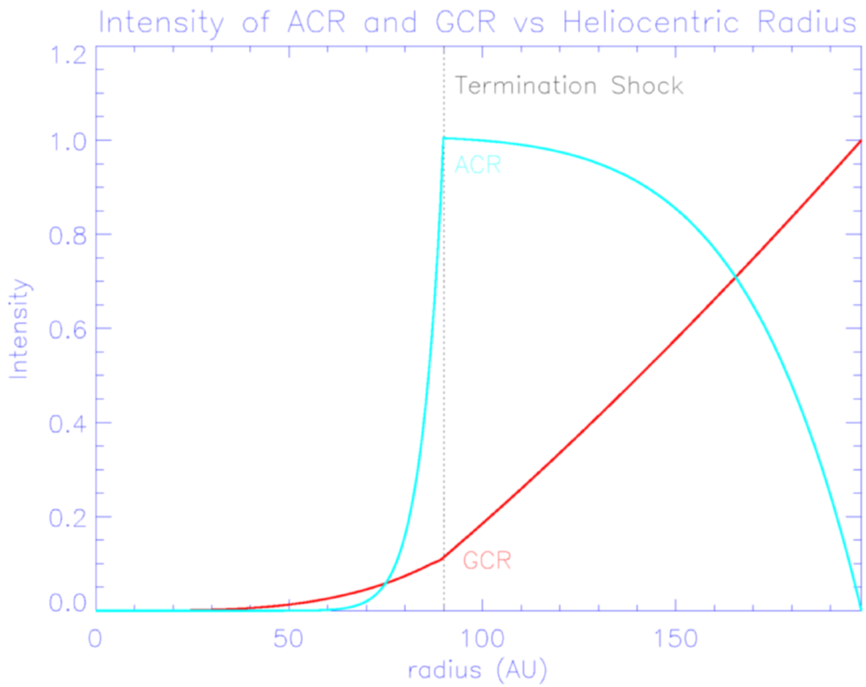

Figure 1: Plots of the solutions to problem 6, where GCR corresponds to the solution to part a., and ACR corresponds to the solution to part b.

$$f = A\left[1-\left(\frac{R_{sh}}{R_b}\right)^{\eta}\right]\left(\frac{r}{R_{sh}}\right)^{\alpha_2} \qquad r < R_{sh}$$

In this case γ is is to be determined by the jump condition at the shock.

We again apply the jump condition again by integrating from just inside of the termination shock to just outside. The jump condition which determines γ is

$$\frac{V_w}{\kappa_0} - \alpha + \frac{(V_2 - V_w)}{3\kappa_0}\gamma = 0$$

or

$$\gamma = \frac{3\kappa_0\alpha}{V_2 - V_w} - V_w - \frac{V_w}{V_2 - V_w}$$

The general nature of the solution is given in figure 1.

7. We consider each part separately.

   (a) Qualitatively, we can say that for $A > 0$, the cosmic-ray ions drift into the inner heliosphere from both heliospheric poles. In this case, the increasing warp away from sunspot minimum has little effect. The intensity will decrease only gradually until far from minimum, increased solar activity will decrease the intensity dramatically.

   On the other hand, for $A < 0$, the cosmic-ray ions tend to come into the inner heliosphere along the current sheet. Hence, as its warp increases away from sunspot minimum, there will be an immediate effect on the cosmic-ray intensity.

(b) For an extended period of low solar activity, as at the Maunder minimum, one would expec that the drift motions would dominate. Hence one expects smaller modulation. But there will still be some, and this modulation would be different for the two signs of the solar magnetic field. This would quite naturally produce an enhanced 22-year variation, which could even dominate the effects of solar activity at the weak solar maximum.

The observations are qualitatively in agreement with this expectation, suggesting that the picture captures a significant part of reality.