# Homework exercise: MHD dynamos

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## 1 Turbulent induction effects

The purpose of this problem set is to derive expressions for the mean field electromotive force $\overline{\boldsymbol{\mathcal{E}}}$ under strongly simplifying assumptions. We start with a few problems in tensor algebra to recall a few mathematical skills useful for the following exercises.

### 1.1 Tensor algebra

For the following mathematical derivations it will be useful to use a component based notation for the manipulation of vector/tensor expressions. A term like $(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}$ can be expressed as $A_k\partial B_i/\partial x_k$, where we use the "summation convention", which assumes that the duplication of the index "k" implies summation $k = 1, 2, 3$. Using the total antisymmetric Levi-Civita tensor $\varepsilon_{ijk}$ we can express a cross product $\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{C}$ as $C_i = \varepsilon_{ijk}A_jB_k$. Note that $\varepsilon_{ijk}$ is +1 for even perturbations of (1,2,3), -1 for odd perturbations of (1,2,3) and 0 otherwise. A useful relation for expressions including products of the Levi-Civita tensor is the identity ("contracted epsilon identity")

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{1}$$

with the Kronecker symbol $\delta_{ik}$, which is 1 for $i = j$ and 0 otherwise.

**Problems:**

a) Compute the double contraction $\varepsilon_{ijk}\varepsilon_{ijl}$.

b) Proof the vector identity

$$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) = -(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{A}\boldsymbol{\nabla}\cdot\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{A}\ . \tag{2}$$

c) Any anti-symmetric tensor, $a_{ij} = -a_{ji}$, has three independent components (i.e. the elements above the diagonal). It can therefore be expressed in terms of a 3-component vector using the Levi-Civita symbol, $a_{ij} = -\epsilon_{ijk}\gamma_k$. Derive an inverse expression given the vector $\gamma_k$ explicitly in terms of $a_{ij}$.

### 1.2 Second order correlation approximation

**Problems:**

a) Start from the induction equation for $\boldsymbol{B}'$ (Volume I, Eq. 3.44):

$$\frac{\partial\boldsymbol{B}'}{\partial t} = \boldsymbol{\nabla}\times\left(\boldsymbol{v}'\times\overline{\boldsymbol{B}} + \overline{\boldsymbol{v}}\times\boldsymbol{B}' - \eta\boldsymbol{\nabla}\times\boldsymbol{B}' + \boldsymbol{v}'\times\boldsymbol{B}' - \overline{\boldsymbol{v}'\times\boldsymbol{B}'}\right)\ , \tag{3}$$

and assume $\overline{\boldsymbol{v}} = 0$, $|\boldsymbol{B}'| \ll |\overline{\boldsymbol{B}}|$ and neglect the contribution from magnetic resistivity. Formally integrate the equation to obtain a solution for $\boldsymbol{B}'$ and derive an expression for $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{v}'\times\boldsymbol{B}'}$. Assume that $\boldsymbol{v}'$ has a finite correlation time, $\tau_c$, and simplify expressions by approximating time integrals with $\int_{-\infty}^{t}\overline{v_i'(t)v_k'(s)}\mathrm{d}s = \tau_c\overline{v_i'(t)v_k'(t)}$.

b) Express now all terms using the component notation summarized in Sect. 1.1 and show that the tensors $a_{ij}$ and $b_{ijk}$ in the expansion $\overline{\mathcal{E}}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$ are given by:

$$a_{ij} = \tau_c \overline{\left( \varepsilon_{ikl} v'_k \frac{\partial v'_l}{\partial x_j} - \varepsilon_{ikj} v'_k \frac{\partial v'_m}{\partial x_m} \right)} \tag{4}$$

$$b_{ijk} = \tau_c \varepsilon_{ijm} \overline{v'_m v'_k} \ . \tag{5}$$

c) Decompose these tensors into the terms $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ defined through:

$$\alpha_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)$$

$$\gamma_i = -\frac{1}{2}\varepsilon_{ijk} a_{jk}$$

$$\beta_{ij} = \frac{1}{4}\left(\varepsilon_{ikl} b_{jkl} + \varepsilon_{jkl} b_{ikl}\right) \ .$$

Compute the trace $\alpha_{ii}$ and $\beta_{ii}$. To which physical quantities are they related?

d) Make now the additional assumption of isotropy, which implies that $\alpha_{ij}$, $\beta_{ij}$, as well as the correlation tensor $\overline{v'_i v'_j}$ are diagonal, i.e. $\alpha_{ij} = \alpha\delta_{ij}$. Compute the scalar $\alpha$-effect and the turbulent diffusivity $\eta_t$. How is $\boldsymbol{\gamma}$ related to $\eta_t$? Discuss under which conditions these effects exist.

## 2 "Biermann battery"

The MHD induction equation is linear in $\boldsymbol{B}$, which implies that a dynamo cannot produce magnetic field if the initial condition was $\boldsymbol{B} = 0$. Start from the more general form of Ohm's law and keep the electron pressure term. Rederive the induction equation and discuss under which conditions the additional term can act as an inhomogeneous source term independent of $\boldsymbol{B}$. Discuss the similarity with the vorticity equation. Describe situations in which this term could have produced weak magnetic seed field in the early universe.