Problem set

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1. Spectra in the IRIS and other wavelength ranges

(a) Identify all spectral features (including any continuum you believe present) in the following spectra of the Sun and the solar-like bright star α Cen A. Explain the rationale why you assign a certain feature to a specific transition (not just a reference). **Resources: Various instruments have observed these wavelengths over many years: SKYLAB SO55, SO62B; OSO-8, SMM-UVSP, HRTS, SUMER, IUE, spectrographs on Hubble.**

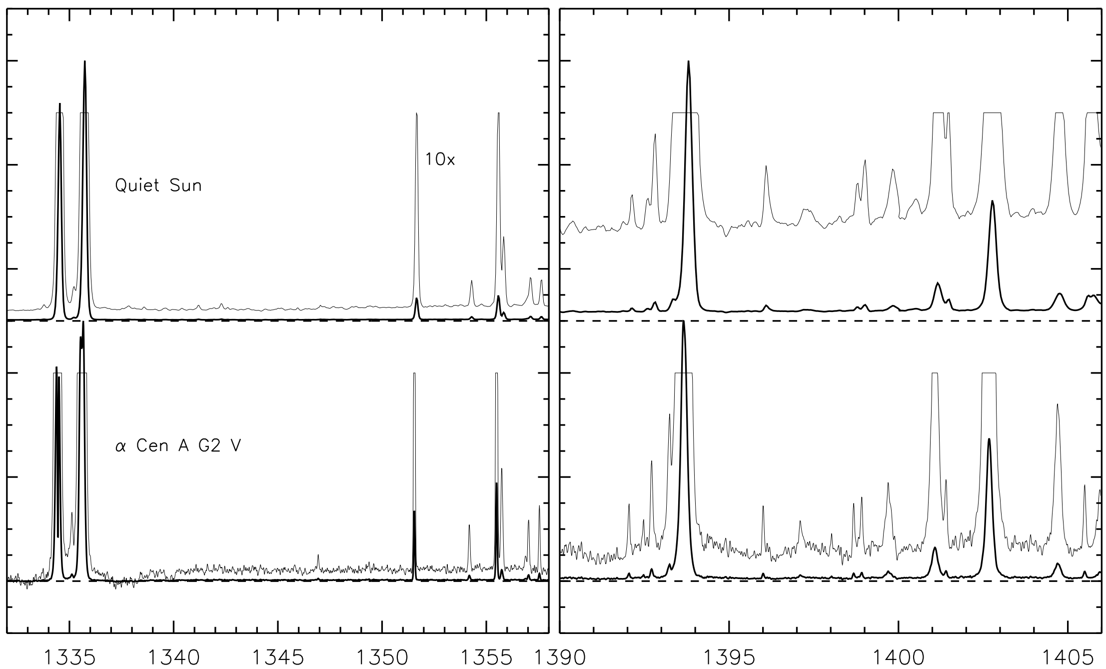

Fig. 1.— UV spectra of the quiet Sun and of α Cen A, obtained with the SUMER instrument on SOHO and the STIS instrument on the Hubble Space Telescope. Only those wavelengths in the far UV observed by IRIS are shown.

(b) There are two kinds of transitions belonging to the *boron isoelectronic sequence* in these data: list these transitions. One of these is an *intercombination* multiplet. How can these "weak" transitions be so visible?

(c) Why do you think there are no “coronal” lines in these spectra (those formed in the coronae where plasma temperatures exceed 10^6K).

(d) How is any continuum formed? How are the brightest lines formed? (Hint: consider broadly such things as the source function, Planck function, the Eddington-Barbier relation, optically thin cases).

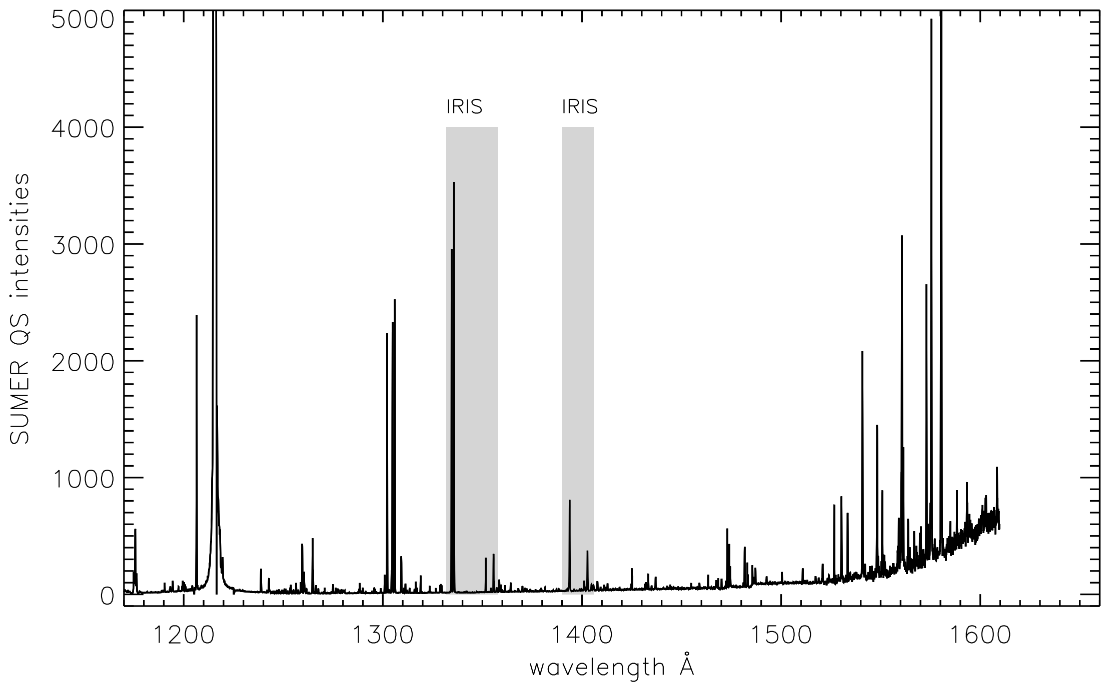

Fig. 2.— The broader UV spectra of Fig. 1.

(e) This figure shows the same UV spectra from 1170 to 1610 Å. Do you think IRIS is missing anything terribly important? If so, what and why?

(f) These are two of the best spectra of their kind. Are you surprised by any differences in the quality of the two datasets? If so or if not, say why.

(g) (Extra credit). Why does the α Cen spectrum have sharp dips in the line near 1334 Å?

2. Transfer equation, formal solutions

(a) Define the specific intensity of radiation I_ν (units: erg $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{hz}^{-1}\,\mathrm{sr}^{-1}$) at frequency ν in terms of energy associated with the passage of a packet of photons across unit area. Explain its invariance properties and physical significance. By assuming that there exist sources and sinks of radiation along a ray, derive the transfer equation along that ray in terms of I_ν, distance along the ray s, an emission coefficient j_ν (units: erg $\mathrm{cm}^{-3}\,\mathrm{s}^{-1}\,\mathrm{hz}^{-1}\,\mathrm{sr}^{-1}$) and absorption coefficient α_ν(units: cm^{-1}).

(b) Define *optical thickness* t_ν in terms of α_ν and element of path length ds, where s measures distance from a point along the ray to the observer. Explain its meaning in terms of the photon mean free path (α_ν represents the probability that a given photon will interact with the material along the ray: in the absence of sources the change dN in the number of photons N along the ray is $dN = -N\alpha_\nu ds$). Consider a path of length S through a tenuous gas with $j_\nu \neq 0$ and $\alpha_\nu \neq 0$ under conditions that $t_\nu \ll 1$. Write down an approximate solution for the emergent intensity $I_\nu(S)$ derived from the above transfer equation along this path. Include any boundary conditions that you think are necessary.

(c) Write down the transfer equation in terms of t_ν and the "source function" $S_\nu = j_\nu/\alpha_\nu$. Explain the physical meaning of S_ν (you might consider using the photon mean free path $\lambda_\nu = \alpha_\nu^{-1}$).

(d) Consider a horizontally homogeneous, vertically stratified atmosphere: all variables are functions of vertical height z only. Let θ be the angle a given ray makes with the vertical direction, $\mu = \cos\theta$, and let $I_{\nu\mu}$ denote the specific intensity along the rays defined by μ. Write down the transfer equation for this ray in terms of the *optical depth* along the vertical rays τ_ν and μ (the optical depth is measured from the observer so it differs in sign from optical thickness). By differentiating $I(x)\,e^{-x}$ with respect to x, and assuming $I_{\nu\mu}(\tau_\nu)\,e^{-\frac{\tau_\nu}{\mu}} \to 0$ as $\tau_\nu \to \infty$, show that the "formal solution" for the intensity along the ray is

$$I_{\nu\mu}(\tau_\nu = 0) = \frac{1}{\mu}\int_0^\infty S_\nu\, e^{-\frac{\tau_\nu}{\mu}}\, d\tau_\nu$$

(e) Assuming that $S_\nu = a + b\tau_\nu$, derive the Eddington-Barbier relation

$$I_{\nu\mu}(\tau_\nu = 0) \;=\; a + b\mu \;=\; S_\nu(\tau_\nu = \mu)$$

and interpret this result in terms of the photon mean free path. Why would this approximation fail for formation under optically thin conditions? Assuming that there is no incoming radiation, ($I_{\nu\mu} = 0$ for all $\mu < 0$), derive similar expressions for the moments: mean intensity $J_\nu(\tau_\nu = 0)$, flux $H_\nu(\tau_\nu = 0)$, and "K-integral" $K_\nu(\tau_\nu = 0)$. Interpret physically these moments in terms of $I_{\nu\mu}(\tau_\nu = 0)$.

(f) Assume you have *optically thin conditions* in a region above the Sun with zero intensity coming up from the Sun's disk. Write down an expression for the intensity of radiation observed along a line of sight intercepting the disk in terms only of the emission coefficient. If we observed slightly above the limb of the Sun, what would happen to the magnitude of the intensity? Now assume that a line of Si IV (3 times ionized silicon) has an emission cofficient integrated over the line at central frequency ν_0 of the form $j_{\nu_0} = \frac{h\nu_0}{4\pi} n_2 A_{21}$ ergs s^{-1}. Write down statistical equlibrium equations for a 2 level atom in terms of coefficients A_{21}, C_{21}, C_{12} (see endnotes). Re-write this in terms of the electron density n_e, the abundance of silicon, the fraction of silicon that is 3x ionized. Lastly, cast the derived intensity into the form containing the emission measure that depends only on atmospheric properties $\xi(T)$ and the "kernel" function $G(T)$ for this specific line and ion.

3. Two level atom in complete redistribution

(a) Consider an atom consisting of just two bound levels, labelled 1 and 2, which have energies E_1, E_2 and degeneracies g_1 and g_2, embedded in a gas in which all particles obey Maxwell-Boltzmann statistics at temperature T. Assume $E_2 > E_1$. The atom can make spontaneous and induced transitions by radiation (Einstein A and B coefficients: see attached "useful formulae"), and by collisions with other particles (e.g. electrons), at total rates given by

$$P_{12} = B_{12}\overline{J} + C_{12} \quad \mathrm{sec}^{-1}, \quad \text{upwards}$$

$$P_{21} = (A_{21} + B_{21}\overline{J}) + C_{21} \quad \mathrm{sec}^{-1}, \quad \text{downwards}$$

Write down an expression for the ratio of the population densities n (cm^{-3}) of the levels, n_1/n_2, under equilibrium conditions.

(b) Construct the source function S_ν, assuming that the only sources of emission and absorption at frequencies in the neighbourhood of the line ($h\nu \simeq E_2 - E_1$), arise from radiative transitions between these levels. Assume that the line is broadened such that it can both absorb and emit radiation in proportion to the same function ϕ_ν where $\int_0^\infty \phi_\nu \, d\nu = 1$. The emission coefficient j_ν and absorption coefficient α_ν are given by:

$$j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi_\nu \ \ \mathrm{erg\ cm^{-3}\,s^{-1}\,hz^{-1}\,sr^{-1}},$$

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \phi_\nu (1 - \frac{n_2 g_1}{n_1 g_2}) \ \ \mathrm{cm}^{-1}.$$

Explain why these coefficients have the forms given. Assume that contributions to ϕ_ν are strongly peaked around the frequency $\nu_0 = (E_2 - E_1)/h$. The source function becomes independent of frequency: it is called the line source function S_L. What is the physical reason that the source function is independent of frequency? [*Hint: you might consider the frequency dependence of the mean free path λ_ν and compare $j_\nu \lambda_\nu$ with $\alpha_\nu \lambda_\nu$.*]

(c) Using the relationships between the A and B coefficients given in the attached "useful formulae", show that S_ν becomes the line source function

$$S_L = \frac{2h\nu_0^3}{c^2} \frac{1}{\frac{n_1 g_2}{n_2 g_1} - 1}.$$

What is the form of S_ν if collisions dominate (n_1/n_2 approaches the Maxwell-Boltzmann ratio), and what does this mean physically?

(d) By combining parts (a) and (c), reduce this expression to the form:

$$S_L = (1-\epsilon)\overline{J} + \epsilon B$$

where

$$\epsilon = \frac{C_{21}(1-e^{-\frac{h\nu_0}{kT}})}{A_{21} + C_{21}(1-e^{-\frac{h\nu_0}{kT}})}$$

[*Hint: eliminate* B_{12}, C_{12} *in favour of* B_{21} *and* C_{21} *in your expression for* n_2/n_1 *in part (a) using the "useful formulae" notes attached to this paper, get a solution for* $\frac{n_1 g_2}{n_2 g_1}$*, and substitute this into the expression for the source function. You should see that terms in* $\overline{J}$ *cancel in the denominator, leaving* $\overline{J}$ *in the numerator only.*] What is the physical meaning of ϵ (it may be simpler initially to ignore the exponential term)?

(e) $\overline{J}$ must be derived from solutions to the transfer equation, which depends on the source function S_L and appropriate boundary conditions. Functionally, this can be written $\overline{J} = \Lambda[S_L]$ where $\Lambda[\ldots]$ denotes the appropriate integral operator. By substituting $\Lambda[S_L]$ for $\overline{J}$ in the equation for S_L derived in part (d), explain what is meant by "lambda-iteration" as a numerical method to try to obtain self-consistent solutions to the resulting integral equation for the source function. [*Hint: write the current estimate of* S_L *at iteration* m *as* $S_L^{(m)}$*, and a new estimate as* $S_L^{(m+1)}$.] Is this method a useful one in general? Explain. Lastly, outline a numerical scheme based upon the concept of "operator splitting": $\Lambda = \Lambda^* + (\Lambda - \Lambda^*)$, to solve this problem, and identify some properties the operator Λ^* should have to make the scheme an attractive one.

4. MHD regimes through the chromosphere.

(a) Magneto-convection generates fields of kG strength in the downflow lanes in the dense sub-photosphere. These concentrations fill only a few percent of the area of the quiet Sun, they are seen as "bright points" in the photosphere, and in the chromosphere they aggregate to form the "chromospheric network". Assume that the chromosphere is stratified hydrostatically (it will also be guided by magnetic structure but if flows/ turbulent motions are subsonic, hydrostatic stratification will be fine) such that

$$p(z) = p_0 e^{-z/h},\ p_0 = 3000 \text{ dyn cm}^{-2},\ h = 150 \text{ km}$$

(These values are typical for a chromospheric temperature $\sim$ 7000K). Write down the height where the magnetic and thermal pressures are the same for field strengths of 1000, 100, 10 G.

(b) The *electron* pressure $p_e = n_e kT_e \sim 0.1$ dyne cm^{-2} is roughly constant throughout the chromosphere, but the total pressure ($p(z) = \sum_i n_i kT$) varies as given above. Throughout most of the chromosphere the proton density $n_p \approx n_e$. (Why?) Given a proton-neutral collision cross section of $\sigma_{pn} \sim 10^{-15}$ cm^2, plot the magnetization of protons $= \omega_p \tau_{pn}$ between $z = 0$ and $z = 1500$ km, where $\tau_{pn} = \langle v\sigma_{pn} \rangle n_n$ is the Maxwellian-averaged collision time for a proton to encounter a neutral hydrogen atom, for a field strength of 1000G.

(c) The magnetic field will expand horizontally with height z to form a "canopy". The expansion is determined by the balance of forces including the $\mathbf{j} \times \mathbf{B}$ Lorentz force so that generally $\mathbf{j} \neq \mathbf{0}$. The magnetic field energy density $\sim B^2$ is expected to fall off less rapidly with height z than $p(z)$. (Why?) Assume $B(z) = 1000 exp(-z/H)$ with $H \sim 300$ km and $B = |\mathbf{B}|$. Again plot the magnetization of protons $= \omega_p \tau_{pn}$ between $z = 0$ and $z = 1500$ km, using $B(z)$. Will ion-neutral collisions serve to damp out ordered motions of the bulk fluid via friction (heating?). If so, where?

(d) Using $B(z)$, plot the Alfvén speed $c_A(z)$ versus z between $z = 0$ and $z = 1500$ km, and over-plot the sound speed for an adiabatic gas with a mean molecular weight (per H atom) of 1.4. Identify places where you might expect sound and MHD waves to interact. Plot the ratio β of magnetic to plasma pressures. Identify places where hydrodynamic stresses likely dominate, and where magnetic stresses dominate. Explain why, if magnetic stresses dominate, the "force-free field" limit of $\mathbf{j} \times \mathbf{B} \approx \mathbf{0}$ is a useful approximation. Will the corona be expected to be force-free?

5. Transport in a partially ionized atmosphere.

(a) **Needed equations** A "simple" hydrogen plasma consists of ions, electrons and neutrals described using fluid equations. Assume neutrality ($n_i = n_e$), neglect, for simplicity, viscosity, thermal forces and anisotropy of the coefficient of friction in a magnetic field. The equations of motion for the electrons, ions and neutrals are

$$-m_e n_e \frac{\mathrm{d}\mathbf{u}_\mathrm{e}}{\mathrm{dt}_e} - \nabla p_e + m_e n_e \mathbf{g} - en\left\{\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right\} = -\alpha_e \frac{\mathbf{j}}{en} + \alpha_{en}\mathbf{w}, \tag{1}$$

$$-m_i n_i \frac{\mathrm{d}\mathbf{u}_\mathrm{i}}{\mathrm{dt}_i} - \nabla p_i + m_i n_i \mathbf{g} + en\left\{\mathbf{E} + \mathbf{u}_i \times \mathbf{B}\right\} = \alpha_{ep} \frac{\mathbf{j}}{en} + \alpha_{in}\mathbf{w}, \tag{2}$$

$$-m_n n_n \frac{\mathrm{d}\mathbf{u}_\mathrm{n}}{\mathrm{dt}_n} - \nabla p_n + m_n n_n \mathbf{g} = \alpha_{en} \frac{\mathbf{j}}{en_e} - \alpha_n\mathbf{w}, \tag{3}$$

The frictional forces $\mathbf{R}_{ab}$ between different particles a, b are described in terms of coefficients α,

$$\mathbf{R}_{ab} = -\alpha_{ab}(\mathbf{u}_a - \mathbf{u}_b) \tag{4}$$

which have the properties

$$\alpha_{ab} = \alpha_{ba},$$

$$\alpha_e = \alpha_{ep} + \alpha_{en} = m_{ep}n/\tau_{ep} + m_{en}n/\tau_{en} = m_e n/\tau_e, \quad \tau_e^{-1} = \tau_{ep}^{-1} + \tau_{en}^{-1},$$

$$\alpha_n = \alpha_{en} + \alpha_{in} = \frac{m_n n_n}{\tau_n}, \quad \tau_n^{-1} = \tau_{ni}^{-1} + \tau_{ne}^{-1},$$

where $n_e = n_i = n$. τ values are average collision times. Collisions with neutrals are dominated by those with ions, because momentum transfer is inefficient between particles of very different mass. Thus

$$\tau_{en} \gg \tau_{in}, \quad \alpha_{en} \ll \alpha_{in}, \text{and} \quad \epsilon = \alpha_{en}/\alpha_n \ll 1.$$

(b) Manipulate the momentum equations such that expressions for center of mass motion of the fluid, the electric current density, and the "ambipolar" flow are obtained in place of the individual momentum equations. NOTE: In this way the coupling between the plasma and the electromagnetic field is made clearer, since current density $\mathbf{j}$ is a source term for the magnetic field $\mathbf{B}$ and the Lorentz force $\mathbf{j} \times \mathbf{B}$ enters into the bulk equation of motion. (The center of mass flow velocity $\mathbf{v}$, the electric current $\mathbf{j}$, and the ion-neutral drift ("ambipolar") velocity $\mathbf{w}$, are simply

$$\mathbf{v} = \frac{\rho_e \mathbf{u}_e + \rho_i \mathbf{u}_i + \rho_n \mathbf{u}_n}{\rho} \approx \frac{\rho_i \mathbf{u}_i + \rho_n \mathbf{u}_n}{\rho}, \tag{5}$$

$$\mathbf{j} = -en_e(\mathbf{u}_e - \mathbf{u}_i), \tag{6}$$

$$\mathbf{w} = \mathbf{u}_i - \mathbf{u}_n. \tag{7}$$

Here, $\rho_e = nm_e$, $\rho_i = nm_i$, $\rho_i = n_n m_n$, $\rho = \rho_e + \rho_i + \rho_n \approx \rho_i + \rho_n$, and $\xi_n = \frac{\rho_n}{\rho}$ is the neutral fraction. Here we have ignored terms of order m_e/m_i.)

Derive the bulk equation of motion from the individual equations:

$$-\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}} - \nabla p + \rho\mathbf{g} + \mathbf{j}\times\mathbf{B} = 0 \tag{8}$$

Following Braginskii, solve for $\mathbf{w}$ and $\mathbf{j}$ from equations (1) - (3). Braginskii notes that when ion-neutral drift accelerations occur more slowly than those for the collisions, then terms of order $d\mathbf{w}/dt$ can be neglected compared with the collisional terms $\mathbf{w}/\tau$ in the frictional force. Then, with $\frac{d}{dt}_i\mathbf{u}_i \approx \frac{d}{dt}_n\mathbf{u}_n \approx \frac{d}{dt}\mathbf{u}$, show that

$$\mathbf{w} = \frac{1}{\alpha_n}(-\mathbf{G} + \xi_n\mathbf{j}\times\mathbf{B}) + \frac{\epsilon\mathbf{j}}{en}, \quad \text{where} \tag{9}$$

$$\mathbf{G} = \xi_n\nabla(p - p_n) - \xi_i\nabla p_n \equiv \xi_n\nabla p - \nabla p_n, \tag{10}$$

where $p = p_e + p_i + p_n$ is the total pressure[1].

Eliminate $\mathbf{w}$ in favor of $\mathbf{G}$ and $\mathbf{j}$, the equation for $\mathbf{j}$, "Ohm's Law", is:

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} + \frac{1}{en}(\nabla p_e - \epsilon\mathbf{G}) - \frac{\xi_n}{\alpha_n}\mathbf{G}\times\mathbf{B} = \frac{1}{\sigma}\mathbf{j} + \frac{\xi_n^2 B^2}{\alpha_n}\mathbf{j}_\perp + \frac{1 - 2\epsilon\xi_n}{en}\mathbf{j}\times\mathbf{B}, \tag{11}$$

where $\mathbf{j}_\perp = \mathbf{b}\times(\mathbf{j}\times\mathbf{b}) = \mathbf{j} - \mathbf{j}_\|$, with $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ and $\mathbf{j}_\| = (\mathbf{b}\cdot\mathbf{j})\mathbf{b}$.

NOTE: The three fluid equations of motion (equations 1 - 3) have been replaced with the bulk equation of motion (8), the equation for ambipolar diffusion (9), and Ohm's law (11). Note that the latter two equations contain pressure gradient terms as well as electrodynamic terms $\mathbf{E} + \mathbf{v}\times\mathbf{B}$, $\mathbf{j}$. Pressure gradient terms do not appear in the conductivity (σ, a scalar or tensor) in the traditional form for Ohm's law $\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B})$.

(c) **Energy dissipation** To order v^2/c^2, the transformation equations of MHD include

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B} \tag{12}$$

$$\mathbf{B}' = \mathbf{B} \tag{13}$$

$$\mathbf{j}' = \mathbf{j} \tag{14}$$

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[1]In the presence of gravity, a term $\frac{m_e}{m_i}\frac{\rho_n\rho_i}{\rho_n+\rho_i}\mathbf{g}$ should be added to the rhs of equation (10), but it is neglected here since it is of order $m_e/m_i\rho\mathbf{g} \ll$ pressure gradients and Lorentz forces.

From Poynting's theorem, the rate at which EM energy is converted to heat is given by $\mathbf{E}' \cdot \mathbf{j}$. Eliminating $\mathbf{E}'$ from equation (11), and neglect terms in $\mathbf{G}$ and p_e in equation (11),

$$\mathbf{E}' \cdot \mathbf{j} = \frac{\mathbf{j}_\parallel^2}{\sigma} + \frac{\mathbf{j}_\perp^2}{\sigma_\perp^*} = \frac{j_\parallel^2 + j_\perp^2}{\sigma} + \frac{\xi_n^2}{\alpha_n} (\mathbf{j} \times \mathbf{B})^2 \,. \tag{15}$$

Show that the conductivities are

$$\sigma = \frac{e^2 n \tau_e}{m_e}, \quad \frac{1}{\sigma_\perp^*} = \frac{1}{\sigma} + \frac{\xi_n^2 B^2}{\alpha_n} = \frac{1}{\sigma}(1 + 2\xi_n \omega_e \tau_e \omega_i \overline{\tau_i}) \tag{16}$$

Where $\overline{\tau_i} = \frac{n_i n_n m_n}{2\alpha_{in}(n_i + n_n)}$. $\sigma_\perp^*$ is Cowling's conductivity: when the plasma has a significant neutral fraction $\xi_n \neq 0$, and when the electrons and/or ions are magnetized ($\omega\tau \gg 1$), the perpendicular conductivity is far smaller than the parallel component σ. Such conditions are found in the solar chromosphere in particular. The photosphere has small values of τ and the corona has $\xi_n \to 0$, both making $\sigma_\perp^* \to \sigma$.

NOTE: While σ depends only on properties of particle impacts, $\sigma_\perp^*$ also depends on the three momentum equations and accompanying assumptions. $\sigma_\perp^*$ is frequently used to describe the dissipation rates by friction in partially ionized plasmas, because one need only evaluate the electric current $\mathbf{j}$ and some thermal and magnetic plasma properties (ionization fractions, collision times, gyrofrequencies) to obtain the energy dissipation, which can be obtained, for example from a single fluid MHD calculation.

NOTE: The above energy dissipation rate originates from evaluating the rate of change of EM energy according to Poynting's theorem. Energy conservation implies that such energy enters the plasma through the dissipation of ordered motions via frictional (and, if present, viscous) forces. Braginskii therefore evaluates dissipation of the directed differential motions represented by $\mathbf{w}$ and $\mathbf{j}$ by explicit evaluation of the total heat generated by friction:

$$\begin{aligned} Q_f &= -\mathbf{R}_{ei} \cdot \mathbf{u} - \mathbf{R}_{en} \cdot (\mathbf{u} + \mathbf{w}) - \mathbf{R}_{in} \cdot \mathbf{w} && (17) \\ &= \alpha_e u^2 + \alpha_n w^2 + 2\alpha_{en} \mathbf{u} \cdot \mathbf{w} \\ &= \frac{j_\parallel^2 + j_\perp^2}{\sigma} + \frac{1}{\alpha_n} (\xi_n \mathbf{j} \times \mathbf{B} - \mathbf{G})^2 \,. && (18) \end{aligned}$$

as expected. (A term of order $\epsilon^2 \approx m_e/m_p$ has been omitted here. Setting $\mathbf{G} = 0$ (i.e. ignoring pressure gradients and gravity) we obtain the same result (equation 15) obtained by evaluation of $\mathbf{E}' \cdot \mathbf{j}$, as we should, under the assumed conditions (collision times $\ll$ dynamic times).)

RADIATION FIELD QUANTITIES, PLANE PARALLEL ATMOSPHERE

μ :	Cosine of the angle a ray makes with the vertical direction
$I_{\nu\mu}$:	Specific intensity at frequency ν along rays defined by μ (erg $\text{cm}^{-2}\,\text{s}^{-1}\,\text{hz}^{-1}\,\text{sr}^{-1}$)
j_ν :	Emission coefficient (erg $\text{cm}^{-3}\,\text{s}^{-1}\,\text{hz}^{-1}\,\text{sr}^{-1}$)
α_ν :	Absorption coefficient (cm^{-1})
$S_\nu = j_\nu/\alpha_\nu$:	Source function (erg $\text{cm}^{-2}\,\text{s}^{-1}\,\text{hz}^{-1}\,\text{sr}^{-1}$)
$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$:	Planck function at temperature T (erg $\text{cm}^{-2}\,\text{s}^{-1}\,\text{hz}^{-1}\,\text{sr}^{-1}$)
$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_{\nu\mu}\,d\mu$:	Mean intensity (erg $\text{cm}^{-2}\,\text{s}^{-1}\,\text{hz}^{-1}\,\text{sr}^{-1}$)
$H_\nu = \frac{1}{2}\int_{-1}^{+1} I_{\nu\mu}\mu\,d\mu$:	Flux
$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_{\nu\mu}\mu^2\,d\mu$:	K-integral
ϕ_ν :	Absorption and emission line profile for an atomic transition (hz^{-1})
$\phi_\nu = \delta(\nu_0 - \nu)$:	Coherent scattering at frequency ν_0
$\int_0^\infty \phi_\nu\,d\nu = 1$:	Normalization for line profile (both absorption and emission)
$\overline{J} = \int_0^\infty J_\nu\,\phi_\nu\,d\nu$:	Frequency-averaged mean intensity (erg $\text{cm}^{-2}\,\text{s}^{-1}\,\text{hz}^{-1}\,\text{sr}^{-1}$)
$\int_0^\infty x^n\,e^{-x}\,dx = n!$	Useful integral, for all non-negative integers n, $0! = 1$

The following apply to transitions by collisions (C) and radiation (Einstein A and B coefficients) between two atomic levels (labelled 1 and 2) of degeneracies g_1 and g_2. Collisions are assumed to arise from impacts by particles with a Maxwell-Boltzmann distribution function at temperature T.

$E_2 > E_1$	Level 2 lies above level 1
$h\nu_0 = E_2 - E_1$:	ν_0 is central frequency of the transition, where h is Planck's constant
$C_{21} \propto n_i\Gamma_{21}(T)$:	$2 \rightarrow 1$ collisional rate by collisions with particles of density n_i ($\Gamma_{21}(T)$ is usually constant or a weak function of T)
A_{21} :	Spontaneous radiative $2 \rightarrow 1$ transition rate
$B_{21}\overline{J}$, $[B_{12}\overline{J}]$:	Induced radiative $2 \rightarrow 1$ $[1 \rightarrow 2]$ transition rate

$\frac{C_{12}}{C_{21}} = \frac{g_2}{g_1} exp\left(-\frac{h\nu_0}{kT}\right)$: ratio of upward to downward collisional transition rates

$g_2 B_{21} = g_1 B_{12}$: Ratio of Einstein B coefficients

$\frac{A_{21}}{B_{21}} = \frac{2h\nu_0^3}{c^2}$: Ratio of Einstein A and B emission coefficients.