SOLUTIONS

1. The upstream deHoffmann-Teller velocity is given by

$$\vec{V}_{HT} = -\frac{\vec{n} \times (\vec{B} \times \vec{V}_{in})}{\vec{n} \cdot \vec{B}}$$

Show that this is also the de Hoffmann-Teller velocity in the region downstream, i.e., that automatically the downstream flow is field-aligned when transforming into the upstream de Hoffmann-Teller frame.

Solution:
$\vec{n} \times (\vec{B} \times \vec{V}_{in})$ is the tangential electric field. The tangential electric field is constant through the shock and does not change. $\vec{n} \cdot \vec{B}$ is the normal magnetic field component. The normal magnetic field component does also not change through the shock. Thus the de Hoffmann-Teller velocity upstream and downstream is the same.

2. Derive the following expression for the ratio of downstream to upstream tangential magnetic field component through a MHD discontinuity

$$\frac{(B_t)_2}{(B_t)_1} = r\frac{(v_n^2)_1 - (c_{int}^2)_1}{(v_n^2)_1 - r(c_{int}^2)_1}$$

where $r = (v_n)_1/(v_n)_2 = \rho_2/\rho_1$ is the compression ratio and $c_{int} = (B_n)_1/(\rho_1\mu$ the upstream intermediate speed. Use for the derivation the tangential momentum jump condition and the condition that the tangential electric field is constant through the shock.

Solution:
The tangential electric field in the HT frame is zero on both sides

$$v_{x1}B_{y1} = v_{y1}B_{x1}; \quad v_{x2}B_{y2} = v_{y2}B_x$$

Note that $B_{x1} = B_{x2} = B_x$

The tangential momentum balance reads:

$$\rho_1 v_{x1} v_{y1} - \frac{B_x B_{y1}}{\mu_0} = \rho_2 v_{x2} v_{y2} - \frac{B_x B_{y2}}{\mu_0}$$

or after substituting for v_{y1} and v_{y2} from the tangential electric field equation

$$\rho_1 v_{x1}^2 \frac{B_{y1}}{B_x} - \frac{B_x B_{y1}}{\mu_0} = \rho_2 v_{x2}^2 \frac{B_{y2}\{B_x}{-} \frac{B_x B_{y2}}{\mu_0}$$

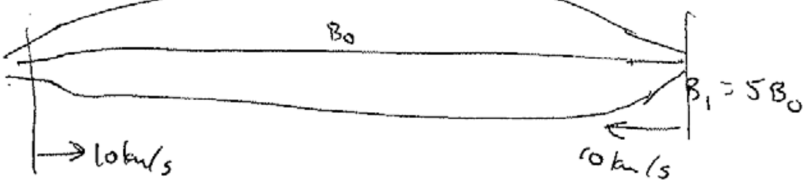
or

$$\frac{B_{y2}}{B_{y1}} = \frac{\rho_1 v_{x1}^2 - B_x/\mu_0}{\rho_2 v_{x2}^2 - B_x/\mu_0}$$

Now use $v_{x2} = v_{x1}\rho_1/\rho_2$ and $\rho_2/\rho_1 = r$
the after some manipulation one can write

$$\frac{(B_t)_2}{(B_t)_1} = r\frac{(v_n^2)_1 - (c_{int}^2)_1}{(v_n^2)_1 - r(c_{int}^2)_1}$$

#3



a)

At midplane $W_0 = W_{\parallel 0} + W_{\perp 0}$ but $\frac{W_{\perp 0}}{B_0} = \mu_m$ is conserved, so energy increase changes $W_{\parallel 0}$ and hence pitch angle. Particle escapes system when field $B_1 = 5B_0$ can't mirror it, ie $B_1 < B_{mirror} = \frac{B_0}{\sin^2\alpha_0}$

So leaves when $5B_0 = \frac{B_0}{\sin^2\alpha_0}$ or $\sin^2\alpha_0 = \frac{1}{5}$

Now $\sin^2\alpha_0 = \frac{W_{\perp 0}}{W_0(t)} = \frac{\frac{1}{2}W_0(t=0)}{W_0(t)}$ since initially $W_{\perp 0} = W_{\parallel 0}$

Thus escapes when $\frac{1}{5} = \frac{\frac{1}{2}W_0(t=0)}{W_0(t)} \Rightarrow W_0(t_{esc}) = \frac{5}{2}W_0(t=0) = \frac{5}{2}\times 1\,\text{keV}$

ie escapes when $\underline{\underline{W = 2.5\,\text{keV}}}$

b) In rest frame of mirror, particle reverses ∥ motion & conserves energy. Transforming to lab frame shows at each bounce the parallel speed has increased by $2V_m$, ie $\Delta v_\parallel = 2V_m n$ where n is the number of bounces. The total change required can be found from:

$$W_{\parallel 0}(i) = W_{\perp 0}(i) = 0.5\,\text{keV} \quad \text{and} \quad W_{\perp 0}(\text{final}) = W_{\perp 0}(i), \text{ so}$$

$$W_{\parallel 0}(f) = W - W_{\perp 0} = 2.5 - 0.5 = 2.0\,\text{keV}$$

Thus $v_{\parallel 0}(i) = \sqrt{\frac{2}{m}0.5\,\text{keV}}$; $v_{\parallel 0}(f) = \sqrt{\frac{2}{m}2.0\,\text{keV}}$

$$\text{so } \Delta v_\parallel(\text{tot}) = v_{\parallel 0}(f) - v_{\parallel 0}(i) = \sqrt{\frac{2}{m}1\,\text{keV}}\left(\sqrt{2} - \sqrt{\tfrac{1}{2}}\right) = \sqrt{\frac{2}{m}1\,\text{keV}}\;\frac{\sqrt{2}}{2}$$

For proton $\sqrt{\frac{2}{m}1\,\text{keV}} = \sqrt{\frac{2\times10^3\times1.6\times10^{-19}}{1.7\times10^{-27}}} = \sqrt{2\times10^{11}} = 4.5\times10^5\ \text{m/s}$

Thus $\Delta v_\parallel(\text{tot}) = 4.5\times10^5\times0.7 = 2\times10\times10^3\,n \Rightarrow n \approx \frac{3\times10^5}{2\times10^4} \sim \underline{15 \text{ bounces}}$

Quick estimate of time needed: Assume $v_{\parallel 0} \sim 5\times10^5$ m/s, $L = 10^{10}$ km $=$ const

$$\text{so } t = \frac{15\times10^{10}\times10^3\,\text{m}}{5\times10^5} = 3\times10^8\,\text{s} \approx 10 \text{ years} \quad [1\text{ yr} \approx 3\times10^7 \text{ secs}]$$

#4 SEE “SHOCKS CHAPTER” on summer school Nicemeeting website

#5

Maxwell eqn $\nabla \cdot \underline{B} = 0$

1-D shock $\frac{\partial}{\partial y} \equiv \frac{\partial}{\partial z} \equiv 0 \Rightarrow \frac{\partial}{\partial x} B_x = 0 \Rightarrow \underline{[B_x] = 0}$

Ideal MHD eqn for mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

steady state, so $\frac{\partial}{\partial t} \equiv 0 \Rightarrow \frac{\partial}{\partial x}(\rho V_x) = 0 \Rightarrow \underline{[\rho V_x] = 0}$

(b) MHD Momentum equation

$$\rho \left\{ \frac{\partial}{\partial t} + \underline{V} \cdot \nabla \right\} \underline{V} = -\nabla p + \frac{1}{\mu_0}(\nabla \wedge \underline{B}) \wedge \underline{B}$$

$$= -\nabla p - \nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\underline{B} \cdot \nabla)\underline{B}$$

$\frac{\partial}{\partial t} \equiv 0$ and $\nabla \equiv \left(\frac{\partial}{\partial x}, 0, 0\right)$ so no transverse components in terms containing $\nabla()$ operator.

$$\Rightarrow (\rho(\underline{V} \cdot \nabla)V)_t = \frac{1}{\mu_0}((\underline{B} \cdot \nabla)\underline{B})_t$$

$$\Rightarrow \rho V_x \frac{\partial}{\partial x} \underline{V}_t = \frac{1}{\mu_0} B_x \frac{\partial}{\partial x} \underline{B}_t$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\rho V_x \underline{V}_t - \frac{B_x}{\mu_0} \underline{B}_t\right) = 0$$ since $\frac{\partial}{\partial x} B_x = \frac{\partial}{\partial x} \rho V_x = 0$

$$\Rightarrow \underline{\left[\rho V_x \underline{V}_t - \frac{B_x}{\mu_0} \underline{B}_t\right] = 0} \qquad (1)$$

(c)

$$[V_x \underline{B}_t - B_x \underline{V}_t] = 0 \qquad (2)$$

$$[B_x] = 0 \qquad (3)$$

$$[\rho V_x] = 0 \qquad (4)$$

$$(1) \div \rho V_x \quad \Rightarrow \quad \left[\underline{V}_t - \frac{B_x}{\mu_0 \rho V_x} \underline{B}_t \right] = 0$$

since $[\rho V_x] = 0$ ie ρV_x constant across shock

$$\times B_x \qquad \left[B_x \underline{V}_t - \frac{B_x^2}{\mu_0 \rho V_x} \underline{B}_t \right] = 0$$

since $[B_x] = 0$

$$\Rightarrow \quad [B_x \underline{V}_t] = \left[\frac{B_x^2}{\mu_0 \rho V_x} \underline{B}_t \right]$$

substitute in (2)

$$\left[V_x \underline{B}_t - \frac{B_x^2}{\mu_0 \rho V_x} \underline{B}_t \right] = 0$$

$$\Rightarrow \quad \left[\left(V_x - \frac{B_x^2}{\mu_0 \rho V_x} \right) \underline{B}_t \right] = 0$$

$$\Rightarrow \quad \alpha_u \underline{B}_{ut} - \alpha_d \underline{B}_{ud} = 0$$

ie $\underline{\underline{B}_{dt} = \alpha \underline{B}_{ut}}$ ie upstream and downstream transverse component vectors of $\underline{B}$ are parallel

$\underline{B}_{ut} \parallel \underline{B}_{dt}$, so plane containing $\hat{n}$ and $\underline{B}_{ut}$ (which also contain $\hat{n}$ and $\underline{B}_d$ since $\underline{B}_n \parallel \hat{n}$) is parallel to plane containing $\hat{n}$ and $\underline{B}_{dt}$. Therefore $\hat{n}$, $\underline{B}_u$ and $\underline{B}_d$ are coplanar. $\underline{B}_u \wedge \underline{B}_d$ is perp to plane containing $\underline{B}_u$ and $\underline{B}_d$.

$\Rightarrow \quad \hat{n} \cdot (\underline{B}_u \wedge \underline{B}_d) = 0$ since $\hat{n}$ is in plane containing $\underline{B}_u$ and $\underline{B}_d$.

(d) $\left. \begin{array}{l} \hat{n} \cdot (\underline{B}_u \wedge \underline{B}_d) = 0 \\ \hat{n} \cdot (\underline{B}_u - \underline{B}_d) = 0 \end{array} \right\}$ Both vectors in brackets are $\perp$ to $\hat{n}$

$\Rightarrow (\underline{B}_u \wedge \underline{B}_d) \wedge (\underline{B}_u - \underline{B}_d) = \underline{N}$

is a vector parallel to $\hat{n}$, since both are non-zero and different.

Normalizing gives $\hat{n} = \dfrac{\underline{N}}{|\underline{N}|}$

#6

(a) For a stationary planar system

$$\frac{d}{dx}(\rho V) = 0$$

$$\rho V \frac{dV}{dx} = -\frac{dP}{dx}$$

$$V\frac{dP}{dx} - \kappa \frac{d^2P}{dx^2} + \gamma \frac{dV}{dx} P = 0$$

(b) $\rho V = A,$ mass flux
$\rho V^2 + P = B,$ momentum flux
For the last equation write

$$V\frac{dP}{dx} - \kappa \frac{d^2P}{dx^2} + \gamma \frac{d}{dx}(PV) - \gamma V \frac{dP}{dx} = 0$$

or

$$-(\gamma - 1)V\frac{dP}{dx} - \kappa \frac{d^2P}{dx^2} + \gamma \frac{d}{dx}(PV) = 0$$

or

$$-(\gamma - 1)\rho V^2\frac{dV}{dx} - \kappa \frac{d^2P}{dx^2} + \gamma \frac{d}{dx}(PV) = 0$$

or

$$\rho V \frac{V^2}{2} - \frac{\kappa}{\gamma - 1}\frac{dP}{dx} + \frac{\gamma}{\gamma - 1} PV = C, \text{ energyflux}$$

Note that $P/(\gamma - 1) = \epsilon$ is the energy density of energetic particles.
(c)

$$A = \rho_0 V_0$$

$$B = \rho_0 V_0^2$$

$$C = \frac{1}{2}\rho_0 V_0^3$$

(d)

$$\rho V \frac{V^2}{2} + \frac{\kappa}{\gamma - 1}\rho V \frac{dV}{dx} + \frac{\gamma}{\gamma - 1} V(B - \rho V^2) = C$$

or

$$\frac{2\kappa}{\gamma+1}\frac{dV}{dx} = (V-V_0)\Big(V - \frac{\gamma-1}{\gamma+1}V_0\Big)$$

(e)
Let $z = x(\gamma+1)x/(2\kappa)$

$$dz = \frac{dV}{(V-V_0)\Big(V-\frac{\gamma-1}{\gamma+1}V_0\Big)}$$

$$= \frac{\gamma+1}{2V_0}\Big[\frac{dV}{V-V_0} - \frac{dV}{V-\frac{\gamma-1}{\gamma+1}V_0}\Big]$$

or

$$z = \frac{\gamma+1}{2V_0}\Big[\ln(V_0-V) - \ln\Big(V - \frac{\gamma-1}{\gamma+1}V_0\Big)\Big] + C$$

note $V < V_0$ and $dV/dx < 0$
C describes the position of the structure in x, we take $C = 0$:

$$\frac{V_0}{\kappa}x = \ln\frac{V_0 - V}{V - \frac{\gamma-1}{\gamma+1}V_0}$$

Solve for $V(x)$

$$V = V_0\frac{1+\frac{\gamma-1}{\gamma+1}\exp(V_0x/\kappa)}{1+\exp(V_0x/\kappa)}$$

$$\rho = \rho_0\frac{1+\exp(V_0x/\kappa)}{1+\frac{\gamma-1}{\gamma+1}\exp(V_0x/\kappa)}$$

$$P = \rho_0V_0^2 - \rho V^2 = \rho_0V_0(V_0 - V)$$

$$= \rho_0V_0^2\frac{2}{\gamma+1}\frac{\exp(V_0x/\kappa)}{[1+\exp(V_0x/\kappa)]}$$

The structure represents a strong modified shock by energetic particles. Since $P(x\to-\infty) = 0$, the shock has infinite Mach number and a compression ratio of 4. The energetic particle acceleration provides all the shock dissipation - there is no fluid subshock.

#7

(a) $\underline{E} = -\underline{V} \wedge \underline{B}$, $\underline{V} = (-V, 0, 0)$ $\underline{B} = (0, 0, B)$

$\Rightarrow$ $\underline{E} = (0, -VB, 0)$

(b) $m\frac{d\underline{u}}{dt} = q(\underline{E} + \underline{u} \wedge \underline{B}) = q\left[(0, -VB, 0) + (u_y B, -u_x B, 0)\right]$

$m\frac{du_x}{dt} = q u_y B$ (1)

$m\frac{du_y}{dt} = -qBV - qBu_x$ (2)

$m\frac{du_z}{dt} = 0$ (3)

(3) $\Rightarrow$ $u_z = \text{constant}$, at $t=0$ $\underline{u} = (V, 0, 0)$ ie $u_z = 0$

$\underline{x} = (0, 0, 0)$ and $z = 0$

Diff (1) and subs. (2) and use $\Omega = \frac{qB}{m}$

$\frac{d^2u_x}{dt^2} = \Omega \frac{du_y}{dt} = -\Omega^2 u_x - \Omega^2 V$

CF: $\frac{d^2u_x}{dt^2} + \Omega^2 u_x = 0$ $\Rightarrow$ $u_{x,cf} = A\cos(\Omega t + \phi)$ A, ϕ constant

PI: $u_{x,PI} = -V$

$\Rightarrow$ $u_x = A\cos(\Omega t + \phi) - V$

(1) $\Rightarrow$ $u_y = \frac{1}{\Omega}\frac{du_x}{dt} = -A\sin(\Omega t + \phi)$

Initial conditions: $u_y(0) = 0$ $\Rightarrow$ $\sin\phi = 0$ $\Rightarrow$ $\phi = 0$

$u_x(0) = V$ $\Rightarrow$ $A - V = V$ $\Rightarrow$ $A = 2V$

Velocity: $u_x = 2V\cos\Omega t - V$

$u_y = -2V\sin\Omega t$

$u_z = 0$

Position: $z = 0$ since $z(0) = 0$

$y = \frac{2V}{\Omega}\cos\Omega t + C_1$ C_1 constant

#7 continued

$$y(0) = 0$$

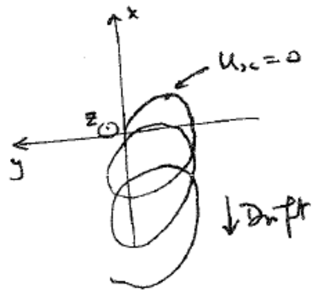
$$\Rightarrow \frac{2V}{\Omega} + c_1 = 0 \quad \Rightarrow c_1 = -\frac{2V}{\Omega}$$

$$\Rightarrow y = \frac{2V}{\Omega}(\cos \Omega t - 1)$$

Integrating: $x = \frac{2V}{\Omega} \sin \Omega t - Vt + c_2$, c_2 constant

But $x(0) = 0 \Rightarrow c_2 = 0$

$$\Rightarrow x = \frac{2V}{\Omega} \sin \Omega t - Vt$$



(c) - No motion in z direction
- uniform drift in x direction at speed V
- gyration around $\underline{B}$ in x-y plane

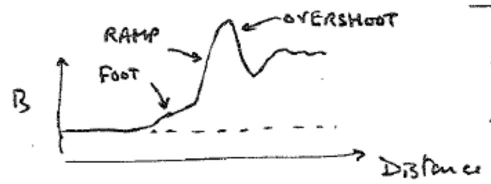
Turning point when $u_x = 0$

$2V \cos \Omega t - V = 0 \Rightarrow \cos \Omega t = \frac{1}{2}$ or $\Omega t = \pi/3$ (60°)

To find x at turning point need $\sin \Omega t$ ie $t = \frac{\pi}{\Omega 3}$

$$\sin \Omega t = \sqrt{1 - \cos^2 \Omega t} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow x_{max} = \frac{2V}{\Omega} \cdot \frac{\sqrt{3}}{2} - \frac{V\pi}{3\Omega} = \frac{V}{\Omega}\left(\sqrt{3} - \frac{\pi}{3}\right)$$

(d) 

FOOT is produced by a fraction of incoming ions (up to 20%) being reflected at the magnetic ramp and gyrating around ahead of the main increase in B.

When downstream the large gyration velocity of the reflected ions contribute to the heating required at the shock for the super- → sub-sonic transition.