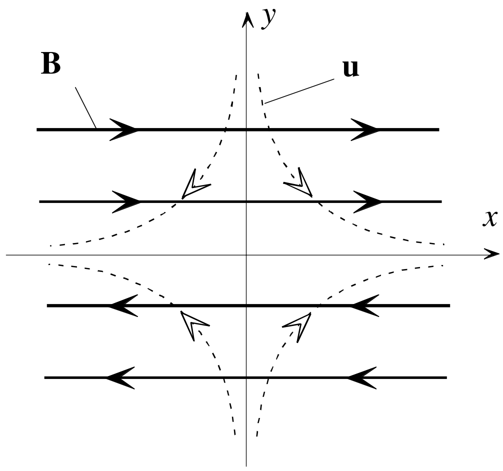
Problems for Magnetic Energy Conversion Processes **T.G. Forbes**

Problem 1. (Vol. 1, Ch. 5). The figure below shows a steady-state configuration where anti-parallel field lines (solid lines) merge and annihilate at the $y = 0$ plane. The annihilation is driven by an imposed stagnation-point flow (dashed lines).



Resistive MHD Equations

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\mathbf{u}\rho) = 0 ,$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\bullet\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{j}\times\mathbf{B} ,$$

$$\nabla\times\mathbf{B} = \mu_0 \mathbf{j} , \quad \mathbf{E} = -\mathbf{u}\times\mathbf{B} + \eta_e \mathbf{j} ,$$

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\mathbf{E} .$$

The stagnation-point flow $\mathbf{u}$ is prescribed by:

$$\mathbf{u} = -ky\,\hat{\mathbf{y}} + kx\,\hat{\mathbf{x}} ,$$

and the magnetic field has the form

$$\mathbf{B}(x, y) = B_x(y)\,\hat{\mathbf{x}} ,$$

where k is a constant and $B_x(y) = -B_x(-y)$.

(a) Use the resistive MHD equations listed above to show that the mass continuity equation is satisfied if the density, ρ, is a constant.

(b) Verify that Faraday's equation is satisfied if the electric field is $\mathbf{E} = -E_o\,\hat{\mathbf{z}}$ where E_o is a constant and $\hat{\mathbf{z}}$ is the unit vector perpendicular to the x-y plane.

(c) Use the resistive MHD Ohm's Law above to determine $B_x(y)$ in terms of the electrical resistivity, η_e, the magnetic permeability, μ_0, and the constants, k and E_O. "Explicitly" means you should solve and integrate the different equation that comes from Ohm's Law. (Hint: You will need to use the Dawson Integral function.)

(d) Plot the solution for B_x with B_x normalized to B_0 and y normalized to l. B_0 and l are constants defined by

$$B_0 = \frac{E_0 \mu_0 l_0}{\eta_e} , \quad \text{and} \quad l_0 = \sqrt{\frac{2\eta_e}{k\mu_0}} .$$

What is the physical significance of B_0 and l_0?

(e) Use the momentum equation to determine the gas pressure $p(x, y)$. What happens to the gas pressure as x or y tends to infinity?

Problem 2. Energy losses due to radiation and thermal conduction play an important role in the dynamics of solar flares. Once a magnetic loop is detached from the reconnection site, it starts to cool. Some simple flare models assume that the cooling occurs at constant pressure so that the density of the plasma, n, is inversely proportional to its temperature, T. These models also assume that the plasma flows in the loop are so slow that the enthalpy transport is negligible. With these assumptions the heat equation that describes the cooling of the plasma along a symmetric flare loop of constant length, L, can be written as:

$$m_i n c_p \frac{\partial T}{\partial t} = \lambda \frac{\partial}{\partial s}\left(T^{5/2} \frac{\partial T}{\partial s} \right) - n^2 \chi T^{\alpha}$$

where m_i is the ion mass, n in the density, c_p is the specific heat at constant pressure, t is time, λ is a constant that measures the strength of the thermal conduction along the loop, s is the distance along the loop, χ is a constant that measures the strength of the radiative loss, and α is a dimensionless constant that describes how the radiative loss varies with temperature. At the top of the loop, $\partial T/\partial s = 0$ because of the imposed symmetry. For the optically thin coronal plasma, α is less than zero in the temperature range between 10^5 K and 10^7 K so that the radiation loss actually decreases as the temperature of the plasma increases. Assume that $\alpha = -1$ and that the temperature at the base of the loops is effectively zero.

(a) Show that if the radiation loss is negligible in the loop, its temperature declines as

$$T = T_0 \left(\frac{s}{L} \right)^{2/7} \left(2 - \frac{s}{L} \right)^{2/7} \left(1 + \frac{7t}{2\tau_{C0}} \right)^{-2/7}$$

where T_0 is the initial temperature at the top of the loop, $s = L$ corresponds to the loop top, $s = 0$ corresponds to the loop footpoint, and τ_{C0} is the linear cooling time prescribed by

$$\tau_{C0} = \frac{7 c_p m_i n_0 L^2}{4 \lambda T_0^{5/2}}$$

where n_0 is the initial density at the top of the loop. Note that τ_{C0} is sometimes referred to as the linear cooling time because it gives the initial rate at which the loops starts to cool when $t/\tau_{C0} << 1$. Plot T/T_0 at the loop top as a function of t/τ_{C0}.

(b) Find the corresponding formula for T as a function of time assuming that the thermal conduction loss is negligible. Express your answer in terms of the linear radiative cooling time, τ_{R0}, and give the formula for τ_{R0}. Plot T/T_0 as a function of t/τ_{R0}.

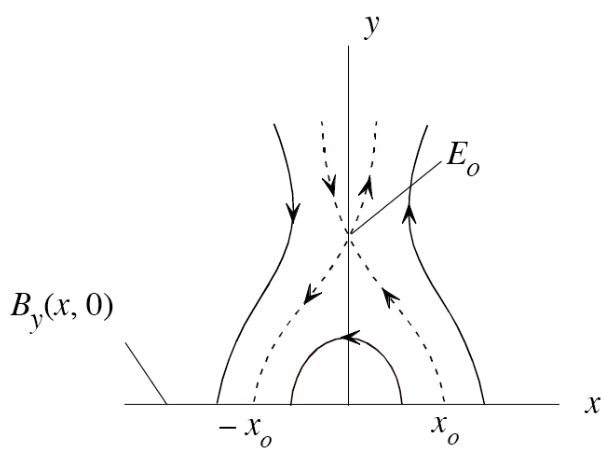
(c) Evaluate τ_{C0} and τ_{R0} at the top of the loop for $L = 10^7$ m, $T_0 = 3 \times 10^7$ K, $n_0 = 10^{16}$ m^{-3}. For the physical constants use $\chi = 2 \times 10^{-29}$ J s^{-1} m^3 K, $\lambda = 10^{-11}$ K$^{-7/2}$ J s^{-1} m^{-1}, m_i =

1.673×10^{-27} kg, and $c_p = 2.064 \times 10^4$ J K^{-1} kg^{-1}. How much slower is the initial radiative cooling time compared to the initial conductive cooling time?

(d) A local, nonlinear cooling time can be defined as:

$$\tau(t) \; = \; -T/(\partial T/\partial t)$$

Use your answers to parts (a) and (b) above to derive formulas for the nonlinear conductive cooling time τ_C and the nonlinear conductive radiative cooling time τ_R at the top of the loop as a function of the temperature ratio T/T_0 and the initial cooling time τ_{C0} or τ_{R0}. If radiative cooling is negligible, how much does the temperature have to decrease by conductive cooling before $\tau_R = \tau_C$? Use your results from part (c) to determine the actual temperature (in degrees k) and time (in seconds) when $\tau_R = \tau_C$ for the flare loop parameters given in part (c).



Problem 3. Consider the above two-dimensional configuration where the field lines at $y = 0$ are anchored in an ideally conducting plate that is stationary (*i.e.* $E_z(x, 0) \; = \; 0$). As reconnection occurs, open field lines are converted to closed loops, and the separatrix footpoints located at $\pm x_o$. appear to move apart. The electric field at the x-line where the separatrices intersect is $E_o \hat{z}$ in the direction perpendicular to the x-y plane.

Use Faraday's equation for an ideal conducting fluid to show that the apparent velocity, $\dot{x}_o$, of the footpoints is given by

$$\dot{x}_0 \; = \; E_0 \; / \; B_y(x_0, 0).$$

Assume that E_o is constant in time and that

$$B_y(x, 0) = B_o \, (a^3 \, x) \, / \, (x^2 + a^2)^2$$

where B_o and a are constants. Determine the footpoint location x_o and its velocity $\dot{x}_o$ as functions of time and sketch their behavior. Assume $x_o = 0$ at $t = 0$.