Creation and destruction of magnetic fields

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Magnetic fields in the Universe

- Earth
  - Magnetic field present for $\sim 3.5 \cdot 10^9$ years, much longer than Ohmic decay time ($\sim 10^4$ years)
  - Strong variability on shorter time scales (10^3 years)
- Mercury, Ganymede, (Io), Jupiter, Saturn, Uranus, Neptune have large scale fields
- Sun
  - Magnetic fields from smallest observable scales to size of sun
  - 11 year cycle of large scale field (Movie)
  - Ohmic decay time $\sim 10^9$ years (in absence of turbulence)
- Other stars
  - Stars with outer convection zone: similar to sun
  - Stars with outer radiation zone: most likely primordial fields
- Galaxies
  - Field structure coupled to observed matter distribution (e.g. spirals)
  - Only dynamo that is directly observable

Scope of this lecture

- Processes of magnetic field generation and destruction in turbulent plasma flows
- Introduction to general concepts of dynamo theory (this is not a lecture about the solar dynamo!)
- Outline
    - MHD, induction equation
    - Some general remarks and definitions regarding dynamos
    - Large scale dynamos (mean field theory)
        - Kinematic theory
        - Characterization of possible dynamos
        - Non-kinematic effects
    - 3D simulations

MHD equations

The full set of MHD equations combines the induction equation with the Navier-Stokes equations including the Lorentz-force:

$$
\begin{aligned}
\frac{\partial \varrho}{\partial t} &= -\boldsymbol{\nabla}\cdot(\varrho\mathbf{v}) \\
\varrho\frac{\partial \mathbf{v}}{\partial t} &= -\varrho(\mathbf{v}\cdot\nabla)\mathbf{v} - \boldsymbol{\nabla}p + \varrho\mathbf{g} + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + \boldsymbol{\nabla}\cdot\boldsymbol{\tau} \\
\varrho\frac{\partial e}{\partial t} &= -\varrho(\mathbf{v}\cdot\nabla)e - p\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T) + Q_\nu + Q_\eta \\
\frac{\partial \mathbf{B}}{\partial t} &= \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B} - \eta\boldsymbol{\nabla}\times\mathbf{B})
\end{aligned}
$$

Assumptions:

- Validity of continuum approximation (enough particles to define averages)
- Non-relativistic motions, low frequencies
- Strong collisional coupling: validity of single fluid approximations, isotropic (scalar) gas pressure

MHD equations

Viscous stress tensor τ

$$
\begin{aligned}
\Lambda_{ik} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \\
\tau_{ik} &= 2\varrho\nu\left(\Lambda_{ik} - \frac{1}{3}\delta_{ik}\nabla\cdot\mathbf{v}\right) \\
Q_\nu &= \tau_{ik}\Lambda_{ik}\ ,
\end{aligned}
$$

Ohmic dissipation Q_η

$$Q_\eta = \frac{\eta}{\mu_0}(\nabla\times\mathbf{B})^2\ .$$

Equation of state

$$p = \frac{\varrho\, e}{\gamma - 1}\ .$$

ν, η and κ: viscosity, magnetic diffusivity and thermal conductivity
μ_0 denotes the permeability of vacuum

Kinematic approach

- Solving the 3D MHD equations is not always feasible
- Semi-analytical approach preferred for understanding fundamental properties of dynamos
- Evaluate turbulent induction effects based on induction equation for a given velocity field
    - Velocity field assumed to be given as 'background' turbulence, Lorentz-force feedback neglected (sufficiently weak magnetic field)
    - What correlations of a turbulent velocity field are required for dynamo (large scale) action?
    - Theory of onset of dynamo action, but not for non-linear saturation

- More detailed discussion of induction equation

Ohm's law

Equation of motion for drift velocity $\mathbf{v}_d$ of electrons

$$m_e \left(\frac{\partial v_d}{\partial t} + \frac{v_d}{\tau_{ei}} \right) = -e(\mathbf{E} + \mathbf{v}_d \times \mathbf{B}) - \nabla p_e$$

τ_{ei}: collision time between electrons and ions
$-e$: electron charge
m_e: electron mass
p_e: electron pressure
With the electric current: $\mathbf{j} = -n\, e\, \mathbf{v}_d$ this gives the generalized Ohm's law:

$$\frac{\partial \mathbf{j}}{\partial t} + \frac{\mathbf{j}}{\tau_{ei}} = \frac{n_e e^2}{m_e}\mathbf{E} - \frac{e}{m_e}\mathbf{j} \times \mathbf{B} + \frac{n_e\, e}{m_e}\nabla p_e$$

Simplifications:

- $\tau_{ei}\, \omega_L \ll 1$, $\omega_L = eB/m_e$: Larmor frequency
- neglect ∇p_e
- low frequencies (no plasma oscillations)

Ohm's law

Simplified Ohm's law

$$\mathbf{j} = \sigma \mathbf{E}$$

with the plasma conductivity

$$\sigma = \frac{\tau_{ei} n_e e^2}{m_e}$$

The Ohm's law we derived so far is only valid in the co-moving frame of the plasma. Under the assumption of non-relativistic motions this transforms in the laboratory frame to

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

Induction equation*

Using Ampere's law $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$ yields for the electric field in the laboratory frame

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\mu_0 \sigma} \boldsymbol{\nabla} \times \mathbf{B}$$

leading to the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B} - \eta\, \boldsymbol{\nabla} \times \mathbf{B})$$

with the magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma} \,.$$

Advection, diffusion, magnetic Reynolds number

L: typical length scale U: typical velocity scale L/U: time unit

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B} - \frac{1}{R_m} \nabla \times \mathbf{B} \right)$$

with the magnetic Reynolds number

$$R_m = \frac{U\,L}{\eta}\ .$$

$R_m \ll 1$: diffusion dominated regime

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B}\ .$$

Only decaying solutions with decay (diffusion) time scale

$$\tau_{\mathrm{d}} \sim \frac{L^2}{\eta}$$

Advection, diffusion, magnetic Reynolds number

$R_m \gg 1$ advection dominated regime (ideal MHD)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Equivalent expression

$$\frac{\partial \mathbf{B}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B}\, \nabla \cdot \mathbf{v}$$

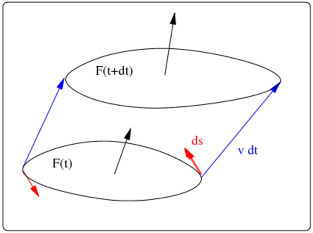
- advection of magnetic field
- amplification by shear (stretching of field lines)
- amplification through compression

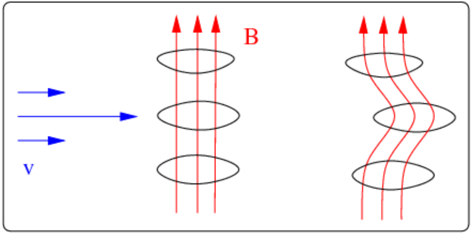
Object	$\eta[\mathrm{m}^2/\mathrm{s}]$	$L[\mathrm{m}]$	$U[\mathrm{m/s}]$	R_m	τ_d
earth (outer core)	2	10^6	10^{-3}	300	10^4 years
sun (plasma conductivity)	1	10^8	100	10^{10}	10^9 years
sun (turbulent conductivity)	10^8	10^8	100	100	3 years
liquid sodium lab experiment	0.1	1	10	100	10 s

Alfven's theorem

Let Φ be the magnetic flux through a surface F with the property that its boundary ∂F is moving with the fluid:

$$\Phi = \int_F \mathbf{B} \cdot d\mathbf{f} \longrightarrow \frac{d\Phi}{dt} = 0$$





- Flux is 'frozen' into the fluid
- Field lines 'move' with plasma

Dynamos: Motivation

- For $\mathbf{v} = 0$ magnetic field decays on timescale $\tau_d \sim L^2/\eta$
- Earth and other planets:
  - Evidence for magnetic field on earth for $3.5 \cdot 10^9$ years while $\tau_d \sim 10^4$ years
  - Permanent rock magnetism not possible since $T > T_{\mathrm{Curie}}$ and field highly variable $\longrightarrow$ field must be maintained by active process
- Sun and other stars:
  - Evidence for solar magnetic field for $\sim 300\,000$ years (^{10}Be)
  - Most solar-like stars show magnetic activity independent of age
  - Indirect evidence for stellar magnetic fields over life time of stars
  - But $\tau_d \sim 10^9$ years!
  - Primordial field could have survived in radiative interior of sun, but convection zone has much shorter diffusion time scale ~ 10 years (turbulent diffusivity)

Mathematical definition of dynamo

S bounded volume with the surface ∂S, $\mathbf{B}$ maintained by currents contained within S

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) && \text{in } S \\
\nabla \times \mathbf{B} &= 0 && \text{outside } S \\
[\mathbf{B}] &= 0 && \text{across } \partial S \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

$\mathbf{v} = 0$ outside S, $\mathbf{n} \cdot \mathbf{v} = 0$ on ∂S and

$$E_{\text{kin}} = \int_S \frac{1}{2} \varrho \mathbf{v}^2 \, dV \leq E_{\max} \quad \forall \, t$$

$\mathbf{v}$ is a dynamo if an initial condition $\mathbf{B} = \mathbf{B}_0$ exists so that

$$E_{\text{mag}} = \int_{-\infty}^{\infty} \frac{1}{2\mu_0} \mathbf{B}^2 \, dV \geq E_{\min} \quad \forall \, t$$

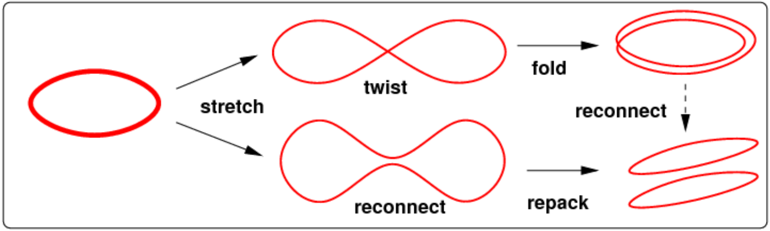
Large scale/small scale dynamos

Decompose the magnetic field into large scale part and small scale part (energy carrying scale of turbulence) $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$:

$$E_{\rm mag} = \int \frac{1}{2\mu_0} \overline{\mathbf{B}}^2 \, dV + \int \frac{1}{2\mu_0} \overline{\mathbf{B}'^2} \, dV \; .$$

- Small scale dynamo: $\overline{\mathbf{B}}^2 \ll \overline{\mathbf{B}'^2}$
- Large scale dynamo: $\overline{\mathbf{B}}^2 \geq \overline{\mathbf{B}'^2}$

Almost all turbulent (chaotic) velocity fields are small scale dynamos for sufficiently large R_m, large scale dynamos require additional large scale symmetries (see second half of this lecture)



- Amplification through field line stretching
- Twist-fold required to repack field into original volume
- Magnetic diffusivity allows for change of topology

Slow/fast dynamos

Influence of magnetic diffusivity on growth rate

- Fast dynamo: growth rate independent of R_m (stretch-twist-fold mechanism)
- Slow dynamo: growth rate limited by resistivity (stretch-reconnect-repack)

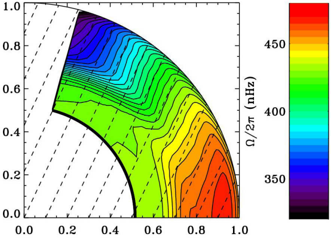
- Fast dynamos relevant for most astrophysical objects since $R_m \gg 1$
- Dynamos including (resistive) reconnection steps can be fast provided the reconnection is fast

Differential rotation and meridional flow

Induction effects of axisymmetric flows on axisymmetric field:

$$
\begin{aligned}
\mathbf{B} &= B\mathbf{e}_{\boldsymbol{\Phi}} + \nabla \times (A\mathbf{e}_{\boldsymbol{\Phi}}) \\
\mathbf{v} &= v_r\mathbf{e}_{\mathbf{r}} + v_\theta\mathbf{e}_{\boldsymbol{\theta}} + \Omega\, r \sin\theta\mathbf{e}_{\boldsymbol{\Phi}}
\end{aligned}
$$

Differential rotation most dominant shear flow in stellar convection zones:



Meridional flow by-product of DR, observed as poleward surface flow in case of the sun

Differential rotation and meridional flow

Spherical geometry:

$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r B) + \frac{\partial}{\partial \theta}(v_\theta B)\right) =$$

$$r \sin \mathbf{B}_p \cdot \nabla\Omega + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) B$$

$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\mathbf{v}_p \cdot \nabla(r\sin\theta A) = \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) A$$

- Meridional flow: Independent advection of poloidal and toroidal field
- Differential rotation: Source for toroidal field (if poloidal field not zero)
- Diffusion: Sink for poloidal and toroidal field
- No term capable of maintaining poloidal field against Ohmic decay!

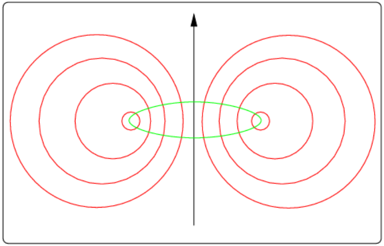
Differential rotation and meridional flow

- Weak poloidal seed field can lead to significant field amplification
- No source term for poloidal field
- Decay of poloidal field on resistive time scale
- Ultimate decay of toroidal field
- Not a dynamo!
- What is needed?

- Source for poloidal field

Cowling's anti-dynamo theorem

A stationary axisymmetric magnetic field with currents limited to a finite volume in space cannot be maintained by a velocity field with finite amplitude.



Ohm's law of the form $\mathbf{j} = \sigma\mathbf{E}$ only decaying solutions, focus here on $\mathbf{j} = \sigma(\mathbf{v} \times \mathbf{B})$.
On O-type neutral line $\mathbf{B}_p$ is zero, but $\mu_0\mathbf{j}_t = \nabla \times \mathbf{B}_p$ has finite value, but cannot be maintained by $(\mathbf{v} \times \mathbf{B})_t = (\mathbf{v}_p \times \mathbf{B}_p)$.

Some history:

- 1919 Sir Joeseph Larmor: Solar magnetic field maintained by motions of conducting fluid?
- 1937 Cowling's anti-dynamo theorem and many others
- 1955 Parker: decomposition of field in axisymmetric and non-axisymmetric parts, average over induction effects of non-axisymmetric field
- 1964 Braginskii, Steenbeck, Krause: Mathematical frame work of mean field theory developed
- last 2 decades 3D dynamo simulations

Reynolds rules

We need to define an averaging procedure to define the mean and the fluctuating field.
For any function f and g decomposed as $f = \overline{f} + f'$ and $g = \overline{g} + g'$ we require that the Reynolds rules apply

$$\begin{aligned}
\overline{\overline{f}} &= \overline{f} \longrightarrow \overline{f'} = 0 \\
\overline{f+g} &= \overline{f} + \overline{g} \\
\overline{f\overline{g}} &= \overline{f}\overline{g} \longrightarrow \overline{f'\overline{g}} = 0 \\
\overline{\partial f/\partial x_i} &= \partial\overline{f}/\partial x_i \\
\overline{\partial f/\partial t} &= \partial\overline{f}/\partial t \, .
\end{aligned}$$

Examples:

- Longitudinal average (mean = axisymmetric component)
- Ensemble average (mean = average over several realizations of chaotic system)

Meanfield induction equation

Average of induction equation:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left(\overline{\mathbf{v}' \times \mathbf{B}'} + \overline{\mathbf{v}} \times \overline{\mathbf{B}} - \eta \nabla \times \overline{\mathbf{B}} \right)$$

New term resulting from small scale effects:

$$\overline{\boldsymbol{\mathcal{E}}} = \overline{\mathbf{v}' \times \mathbf{B}'}$$

Fluctuating part of induction equation:

$$\left(\frac{\partial}{\partial t} - \eta \Delta \right) \mathbf{B}' - \nabla \times (\overline{\mathbf{v}} \times \mathbf{B}') = \nabla \times \left(\mathbf{v}' \times \overline{\mathbf{B}} + \mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'} \right)$$

Kinematic approach: $\mathbf{v}'$ assumed to be given

- Solve for $\mathbf{B}'$, compute $\overline{\mathbf{v}' \times \mathbf{B}'}$ and solve for $\overline{\mathbf{B}}$
- Term $\mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'}$ leading to higher order correlations (closure problem)

Second order correlation approximation (SOCA)

Second order term can be neglected if

- $|\mathbf{B}'| \ll |\overline{\mathbf{B}}|$
- $|\mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'}| \ll |\mathbf{v}' \times \overline{\mathbf{B}}|$
- $\nabla \times (\mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'})$ correlates only weakly with $\mathbf{v}'$

Sufficient condition:

- $R_m \ll 1$ or $S = v\tau_c/l_c \ll 1 \longrightarrow |\mathbf{B}'| \ll |\overline{\mathbf{B}}|$
- Problem: $R_m \gg 1$ and $S \sim 1$ in stellar convection zones

In praxis it works better than it should!

Second order correlation approximation (SOCA)

Neglecting higher order moments and assume $\overline{\tau} \gg \tau_c$:

$$\mathbf{B}' \approx \tau_c \boldsymbol{\nabla} \times \left(\mathbf{v}' \times \overline{\mathbf{B}}\right) = -\tau_c(\mathbf{v}' \cdot \boldsymbol{\nabla})\overline{\mathbf{B}} + \tau_c[(\overline{\mathbf{B}} \cdot \boldsymbol{\nabla})\mathbf{v}' - \overline{\mathbf{B}}\, \boldsymbol{\nabla} \cdot \mathbf{v}']$$

leads to the expression:

$$\overline{\boldsymbol{\mathcal{E}}} = \boldsymbol{\alpha}\overline{\mathbf{B}} + \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta}\, \boldsymbol{\nabla} \times \overline{\mathbf{B}} + \ldots$$

with ($\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are symmetric tensors):

$$\begin{aligned}
\alpha_{ij} &= \frac{1}{2}\tau_c \left(\varepsilon_{ikl}\overline{v_k{}'\frac{\partial v_l{}'}{\partial x_j}} + \varepsilon_{jkl}\overline{v_k{}'\frac{\partial v_l{}'}{\partial x_i}}\right) \\
\gamma_i &= -\frac{1}{2}\tau_c \frac{\partial}{\partial x_k}\overline{v_i' v_k'} \\
\beta_{ij} &= \frac{1}{2}\tau_c \left(\overline{\mathbf{v}'^2}\delta_{ij} - \overline{v_i{}' v_j{}'}\right)
\end{aligned}$$

Second order correlation approximation (SOCA)

Simplification for (quasi) isotropic, non-mirror symmetric, (weakly) inhomogeneous turbulence:

$$\overline{v_i' v_j'} \sim \delta_{ij},\ \alpha_{ij} = \alpha\delta_{ij},\ \beta_{ij} = \eta_t\delta_{ij}$$

Leads to:

$$\begin{aligned}
\alpha &= \frac{1}{3}\alpha_{ii} = -\frac{1}{3}\tau_c\,\overline{\mathbf{v}'\cdot(\boldsymbol{\nabla}\times\mathbf{v}')} \sim \frac{\eta_t}{l_c} \sim v_{rms} \\
\eta_t &= \frac{1}{3}\beta_{ii} = \frac{1}{3}\tau_c\,\overline{\mathbf{v}'^2} \sim l_c\, v_{rms} \\
\gamma &= -\frac{1}{2}\boldsymbol{\nabla}\eta_t
\end{aligned}$$

Induction equation for $\overline{\mathbf{B}}$:

$$\frac{\partial\overline{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla}\times\left[\alpha\overline{\mathbf{B}} + (\overline{\mathbf{v}}+\gamma)\times\overline{\mathbf{B}} - (\eta+\eta_t)\,\boldsymbol{\nabla}\times\overline{\mathbf{B}}\right]$$

Turbulent diffusivity - destruction of magnetic field

Turbulent diffusivity dominant dissipation process for large scale field in case of large R_m:

$$\eta_t = \frac{1}{3}\tau_c\,\overline{\mathbf{v}'^2} \sim L\,v_{\rm rms} \sim R_m\eta \gg \eta$$

- Formally η_t comes from advection term (transport term, non-dissipative)
- Turbulent cascade transporting magnetic energy from the large scale L to the micro scale l_m (advection + reconnection)

$$\eta\mathbf{j}_m^2 \sim \eta_t\bar{\mathbf{j}}^2 \longrightarrow \frac{B_m}{l_m} \sim \sqrt{R_m}\frac{\overline{B}}{L}$$

Important: The large scale determines the energy dissipation rate, l adjusts to allow for the dissipation on the microscale.

Present for isotropic homogeneous turbulence

Turbulent diamagnetism, turbulent pumping

Expulsion of flux from regions with larger turbulence intensity 'diamagnetism'

$$\gamma = -\frac{1}{2}\boldsymbol{\nabla}\eta_t$$

Downward directed at base of convection zone

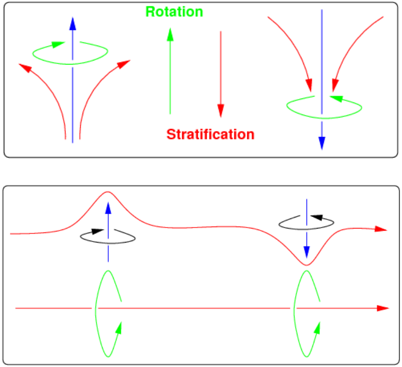
Turbulent pumping (stratified convection):

- Upflows expand, downflows converge
- Stronger velocity and smaller filling factor of downflows
- Mean advection effect of up- and downflows does not cancel
- Downward transport found in numerical simulations

Requires inhomogeneity (stratification)

$$\alpha = -\frac{1}{3}\tau_c \overline{\mathbf{v}' \cdot (\nabla \times \mathbf{v}')} \qquad H_k = \overline{\mathbf{v}' \cdot (\nabla \times \mathbf{v}')} \text{ kinetic helicity}$$

Requires rotation + additional preferred direction (stratification)



Fast or slow dynamo?

Turbulent induction effects require reconnection to operate; however, the expressions

$$\begin{aligned}
\alpha_{ij} &= \frac{1}{2}\tau_c \left(\varepsilon_{ikl} \overline{{v_k}' \frac{\partial {v_l}'}{\partial x_j}} + \varepsilon_{jkl} \overline{{v_k}' \frac{\partial {v_l}'}{\partial x_i}} \right) \\
\gamma_i &= -\frac{1}{2}\tau_c \frac{\partial}{\partial x_k} \overline{v_i' v_k'} \\
\beta_{ij} &= \frac{1}{2}\tau_c \left(\overline{\mathbf{v}'^2} \delta_{ij} - \overline{{v_i}' {v_j}'} \right)
\end{aligned}$$

are independent of η (in this approximation), indicating fast dynamo action.

Validity of Mean field expansion

Second order correlation approximation:

- At best marginally justified
- Works better than it should

Most general form for mean field expansion:

$$\overline{\mathcal{E}}_i(\mathbf{x},t) = \int_{-\infty}^{\infty} d^3x' \int_{-\infty}^{t} dt' \, \mathcal{K}_{ij}(\mathbf{x},t,\mathbf{x}',t') \, \overline{\mathbf{B}}_j(\mathbf{x}',t') \, .$$

Sufficient scale separation

- $l_c \ll L$
- $\tau_c \ll \tau_L$

leads to:

$$\overline{\boldsymbol{\mathcal{E}}} = \boldsymbol{\alpha}\overline{\mathbf{B}} + \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta}\,\nabla \times \overline{\mathbf{B}} - \boldsymbol{\delta} \times \nabla \times \overline{\mathbf{B}} + \ldots$$

In stellar convection zones scale separation also only marginally justified (continuous turbulence spectrum)!

Large scale convection (M. Miesch, HAO)

Symmetry constraints

α, β, γ and δ depend on large scale symmetries of the system defining the symmetry properties of the turbulence (e.g. rotation and stratification). Additional to that the expansion

$$\overline{\boldsymbol{\mathcal{E}}} = \alpha \overline{\mathbf{B}} + \gamma \times \overline{\mathbf{B}} - \beta \nabla \times \overline{\mathbf{B}} - \delta \times \nabla \times \overline{\mathbf{B}} + \dots$$

is a relation between polar and axial vectors:

- $\overline{\boldsymbol{\mathcal{E}}}$: polar vector, independent from handedness of coordinate system
- $\mathbf{B}$: axial vector, involves handedness of coordinate system in definition (curl operator, cross product)

Handedness of coordinate system pure convention (contains no physics), consistency requires:

- α, δ: pseudo tensor
- β, γ: true tensors

Symmetry constraints

Turbulence with rotation and stratification

- true tensors: δ_{ij}, g_i, $g_i g_j$, $\Omega_i \Omega_j$, $\Omega_i \varepsilon_{ijk}$
- pseudo tensors: ε_{ijk}, Ω_i, $\Omega_i g_j$, $g_i \varepsilon_{ijk}$

Symmetry constraints allow only certain combinations:

$$\alpha_{ij} = \alpha_0 (\mathbf{g} \cdot \mathbf{\Omega}) \delta_{ij} + \alpha_1 \left(g_i \Omega_j + g_j \Omega_i \right) , \quad \gamma_i = \gamma_0 g_i + \gamma_1 \varepsilon_{ijk} g_j \Omega_k$$
$$\beta_{ij} = \beta_0 \, \delta_{ij} + \beta_1 \, g_i g_j + \beta_2 \, \Omega_i \Omega_j , \quad \delta_i = \delta_0 \Omega_i$$

The scalars $\alpha_0 \ldots \delta_0$ depend on quantities of the turbulence such as rms velocity and correlation times scale.

- isotropic turbulence: only $\boldsymbol{\beta}$
- $+$ stratification: $\boldsymbol{\beta} + \boldsymbol{\gamma}$
- $+$ rotation: $\boldsymbol{\beta} + \boldsymbol{\delta}$
- $+$ stratification $+$ rotation: $\boldsymbol{\alpha}$ can exist

Generalized Ohm's law

What is needed to circumvent Cowling's theorem?

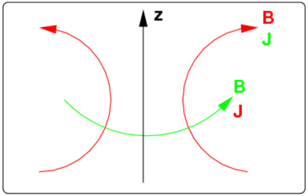
- Crucial for Cowling's theorem: Impossibility to drive a current parallel to magnetic field
- Cowling's theorem does not apply to mean field if a mean current can flow parallel to the mean field (since total field non-axisymmetric this is not a contradiction!)

$$\bar{\mathbf{j}} = \tilde{\boldsymbol{\sigma}} \left(\overline{E} + \bar{\mathbf{v}} \times \overline{\mathbf{B}} + \boldsymbol{\gamma} \times \overline{\mathbf{B}} + \boldsymbol{\alpha} \overline{\mathbf{B}} \right)$$

$\tilde{\boldsymbol{\sigma}}$ contains contributions from η, β and δ.
Ways to circumvent Cowling:

- α-effect
- anisotropic conductivity (off diagonal elements + δ-effect)

α^2-dynamo



Induction of field parallel to current (producing helical field!)

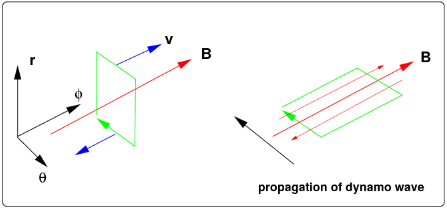
$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\alpha \overline{\mathbf{B}}) = \alpha \mu_0 \bar{\mathbf{j}}$$

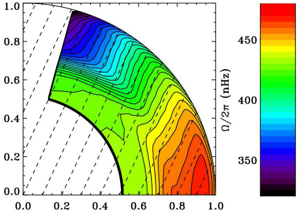
Dynamo cycle:

$$\mathbf{B}_t \xrightarrow{\alpha} \mathbf{B}_p \xrightarrow{\alpha} \mathbf{B}_t$$

- Poloidal and toroidal field of similar strength
- In general stationary solutions

$\alpha\Omega$-, $\alpha^2\Omega$-dynamo





Dynamo cycle:

$$\mathbf{B}_t \xrightarrow{\alpha} \mathbf{B}_p \xrightarrow{\Omega} \mathbf{B}_t$$

- Toroidal field much stronger that poloidal field
- In general traveling (along lines of constant Ω) and periodic solutions

$\alpha\Omega$-dynamo

$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r B) + \frac{\partial}{\partial \theta}(v_\theta B)\right) = r \sin \mathbf{B}_p \cdot \boldsymbol{\nabla}\Omega$$
$$+ \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) B$$
$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\mathbf{v}_p \cdot \boldsymbol{\nabla}(r\sin\theta A) = \alpha B + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) A$$

- Dimensionless measure for strength of Ω- and α-effect

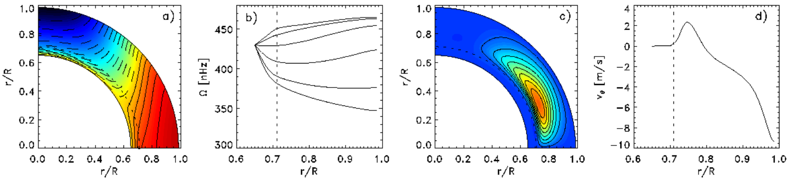
$$D_\Omega = \frac{R^2 \Delta\Omega}{\eta_t} \qquad D_\alpha = \frac{R\alpha}{\eta_t}$$

- Dynamo excited if dynamo number

$$D = D_\Omega D_\alpha > D_{crit}$$

Movie: $\alpha\Omega$-dynamo

$\alpha\Omega$-dynamo with meridional flow



Meridional flow:

- Poleward at top of convection zone
- Equatorward at bottom of convection zone

Effect of advection:

- Equatorward propagation of activity
- Correct phase relation between poloidal and toroidal field
- Circulation time scale of flow sets dynamo period
- Requirement: Sufficiently low turbulent diffusivity

Movie: Flux-transport-dynamo (M. Dikpati, HAO)

$\Omega \times J$ dynamo

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla} \times [\delta \times (\boldsymbol{\nabla} \times \overline{\mathbf{B}})] \sim \boldsymbol{\nabla} \times (\Omega \times \bar{\mathbf{j}}) \sim \frac{\partial \bar{\mathbf{j}}}{\partial z}$$

- similar to α-effect, but additional z-derivative of current
- couples poloidal and toroidal field
- δ^2 dynamo is not possible:

$$\bar{\mathbf{j}} \cdot \overline{\mathcal{E}} = \bar{\mathbf{j}} \cdot (\boldsymbol{\delta} \times \bar{\mathbf{j}}) = 0$$

- δ-effect is controversial (not all approximations give a non-zero effect)
- in most situations α dominates

Dynamos and magnetic helicity

Magnetic helicity (integral measure of field topology):

$$H_m = \int \mathbf{A} \cdot \mathbf{B} \, dV$$

has following conservation law (no helicity fluxes across boundaries):

$$\frac{d}{dt} \int \mathbf{A} \cdot \mathbf{B} \, dV = -2\mu_0 \, \eta \int \mathbf{j} \cdot \mathbf{B} \, dV$$

Decomposition into small and large scale part:

$$\begin{aligned}
\frac{d}{dt} \int \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \, dV &= +2 \int \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \, dV - 2\mu_0 \, \eta \int \overline{\mathbf{j}} \cdot \overline{\mathbf{B}} \, dV \\
\frac{d}{dt} \int \overline{\mathbf{A}' \cdot \mathbf{B}'} \, dV &= -2 \int \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \, dV - 2\mu_0 \, \eta \int \overline{\mathbf{j}' \cdot \mathbf{B}'} \, dV
\end{aligned}$$

Dynamos and magnetic helicity

Dynamos have helical fields:

- α-effect induces magnetic helicity of same sign on large scale
- α-effect induces magnetic helicity of opposite sign on small scale

Saturation process (on scale $\sim R_m \tau_c$):

$$\overline{\mathbf{j}' \cdot \mathbf{B}'} = -\bar{\mathbf{j}} \cdot \overline{\mathbf{B}} \longrightarrow \frac{|\overline{B}|}{|B'|} \sim \sqrt{\frac{L}{l}}$$

$$\overline{\mathbf{j}' \cdot \mathbf{B}'} = -\frac{\alpha \overline{\mathbf{B}}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \bar{\mathbf{j}} \cdot \overline{\mathbf{B}}$$

Time scales:

- Galaxy: $\sim 10^{25}$ years ($R_m \sim 10^{18}$, $\tau_c \sim 10^7$ years)
- Sun: $\sim 10^8$ years
- Earth: $\sim 10^6$ years

Non-kinematic effects

Proper way to treat them: 3D simulations

- Still very challenging
- Has been successful for geodynamo, but not for solar dynamo

Semi-analytical treatment of Lorentz-force feedback in mean field models:

- Macroscopic feedback: Change of the mean flow (differential rotation, meridional flow) through the mean Lorentz-force

$$\overline{\mathbf{f}} = \overline{\mathbf{j}} \times \overline{\mathbf{B}} + \overline{\mathbf{j}' \times \mathbf{B}'}$$

- Mean field model including mean field representation of full MHD equations:
  Movie: Non-kinematic flux-transport dynamo
- Microscopic feedback: Change of turbulent induction effects (e.g. α-quenching)

Microscopic feedback

Feedback of Lorentz force on small scale motions:

- Intensity of turbulent motions significantly reduced if $\frac{1}{2\mu_0}B^2 > \frac{1}{2}\varrho v_{rms}^2$. Typical expression used

$$\alpha = \frac{\alpha_k}{1 + \frac{\overline{\mathbf{B}}^2}{B_{eq}^2}}$$

  with the equipartition field strength $B_{eq} = \sqrt{\mu_0 \varrho} v_{rms}$
- Similar quenching also expected for turbulent diffusivity
- Additional quenching of α due to topological constraints possible (helicity conservation)
  Controversial !

Microscopic feedback

Symmetry of momentum and induction equation $\mathbf{v}' \leftrightarrow \mathbf{B}'$:

$$\begin{aligned} \frac{d\mathbf{v}'}{dt} &= \frac{1}{\mu_0 \varrho}(\overline{\mathbf{B}} \cdot \nabla)\mathbf{B}' + \ldots \\ \frac{d\mathbf{B}'}{dt} &= (\overline{\mathbf{B}} \cdot \nabla)\mathbf{v}' + \ldots \\ \overline{\boldsymbol{\mathcal{E}}} &= \overline{\mathbf{v}' \times \mathbf{B}'} \end{aligned}$$

Strongly motivates magnetic term for α-effect (Pouquet et al. 1976):

$$\alpha = \frac{1}{3}\tau_c \left(\frac{1}{\varrho} \overline{\mathbf{j}' \cdot \mathbf{B}'} - \overline{\boldsymbol{\omega}' \cdot \mathbf{v}'} \right)$$

- Kinetic α: $\overline{\mathbf{B}} + \mathbf{v}' \longrightarrow \mathbf{B}' \longrightarrow \overline{\boldsymbol{\mathcal{E}}}$
- Magnetic α: $\overline{\mathbf{B}} + \mathbf{B}' \longrightarrow \mathbf{v}' \longrightarrow \overline{\boldsymbol{\mathcal{E}}}$

Microscopic feedback

From helicity conservation one expects

$$\overline{\mathbf{j}' \cdot \mathbf{B}'} \sim -\alpha \overline{\mathbf{B}}^2$$

leading to algebraic quenching

$$\alpha = \frac{\alpha_k}{1 + g \, \frac{\overline{\mathbf{B}}^2}{B_{eq}^2}}$$

With the asymptotic expression (steady state)

$$\overline{\mathbf{j}' \cdot \mathbf{B}'} = -\frac{\alpha \overline{\mathbf{B}}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \bar{\mathbf{j}} \cdot \overline{\mathbf{B}}$$

we get

$$\alpha = \frac{\alpha_\mathrm{k} + \frac{\eta_t^2}{\eta} \frac{\mu_0 \bar{\mathbf{j}} \cdot \overline{\mathbf{B}}}{B_\mathrm{eq}^2}}{1 + \frac{\eta_t}{\eta} \frac{\overline{\mathbf{B}}^2}{B_\mathrm{eq}^2}}$$

Microscopic feedback

Catastrophic α-quenching ($R_m \gg 1!$) in case of steady state and homogeneous $\overline{\mathbf{B}}$:

$$\alpha = \frac{\alpha_{\mathrm{k}}}{1 + R_m \frac{\overline{\mathbf{B}}^2}{B_{\mathrm{eq}}^2}}$$

If $\bar{\mathbf{j}} \cdot \overline{\mathbf{B}} \neq 0$ (dynamo generated field) and η_t unquenched:

$$\alpha \approx \eta_t \, \mu_0 \frac{\bar{\mathbf{j}} \cdot \overline{\mathbf{B}}}{\overline{\mathbf{B}}^2} \sim \frac{\eta_t}{L} \sim \frac{\eta_t}{l_c} \frac{l_c}{L} \sim \alpha_k \frac{l_c}{L}$$

- In general α-quenching dynamic process: linked to time evolution of helicity
- Boundary conditions matter: Loss of small scale current helicity can alleviate catastrophic quenching
- Catastrophic α-quenching turns large scale dynamo into slow dynamo

Microscopic feedback

Stationary state reached on time scale $R_m \tau_c$:

- Galaxy: $\sim 10^{25}$ years ($R_m \sim 10^{18}$, $\tau_c \sim 10^7$ years)
- Sun: $\sim 10^8$ years
- Earth: $\sim 10^6$ years
- Universe too young for galaxies to worry about stationary state!
- Sun, geodynamo had enough time too saturate
  - Sun: Possibility that helicity loss through photosphere alleviates quenching
  - Geodynamo: $R_m \sim 300$ not that catastrophic?
- No observational evidence for catastrophic α-quenching, but fundamental question for theory!

3D simulations

Why not just solving the full system to account for all non-linear effects?

- Most systems have $R_e \gg R_m \gg 1$, requiring high resolution
- Large scale dynamos evolve on time scales $\tau_c \ll t \ll \tau_\eta$, requiring long runs compared to convective turn over
- 3D simulations successful for geodynamo
  - $R_m \sim 300$: all relevant magnetic scales resolvable
  - Incompressible system
- Solar dynamo: Ingredients can be simulated
  - Compressible system: density changes by 10^6 through convection zone
  - Boundary layer effects: Tachocline, difficult to simulate (strongly subadiabatic stratification, large time scales)
  - Magnetic structures down to 1000 km most likely important Evolve 5000^3 box over 1000 τ_c!
  - Small scale dynamos can be simulated (for $P_m \sim 1$)

Summarizing remarks

Destruction of magnetic field:

- Turbulent diffusivity: cascade of magnetic energy from large scale to dissipation scale (advection+reconnection)
- Enhances dissipation of large field by a factor R_m

Creation of magnetic field:

- Small scale dynamo (non-helical)
  - Amplification of field on and below scale of turbulence
  - Stretch-twist-fold-(reconnect)
  - Produces non-helical field and does not require helical motions
  - Current research: behavior for $P_m \ll 1$
- Large scale dynamo (helical)
  - Amplification of field on scales larger than scale of turbulence
  - Produces helical field and does require helical motions
  - Requires rotation + additional symmetry direction (controversial $\Omega \times J$ effect does not require helical motions)
  - Current research: catastrophic vs. non-catastrophic quenching