Large-Scale Instabilities

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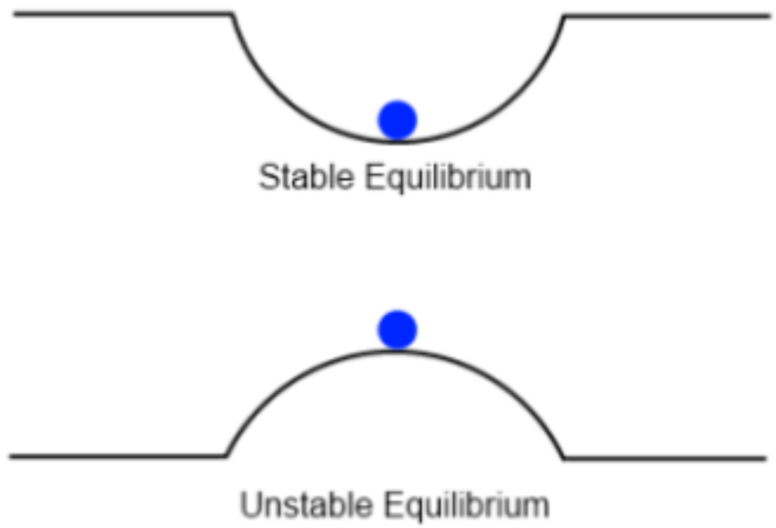
Basic idea behind physical instabilities



- If you perturb this stable equilibrium, the ball will return to its original location, or oscillate around it if damping is weak
- If you perturb this unstable equilibrium, the ball will roll away

Mechanical analogy

- If $W(x)$ is the potential energy, then the particle feeds a force $F_x = -\partial W/\partial x$
- Equilibria occur when $\partial W/\partial x = 0$
- To lowest order in the Taylor expansion about an equilibrium point, the change in potential energy is

$$\delta W = \frac{1}{2}\left(\frac{\partial^2 W}{\partial x^2}\right)(\Delta x)^2 \tag{1}$$

- The equilibria are stable when $\delta W > 0$
- The equilibria are unstable when $\delta W < 0$

Wiki list on plasma instabilities

List of plasma instabilities [edit]

- Bennett pinch instability (also called the z-pinch instability)
- Beam acoustic instability
- Bump-in-tail instability
- Buneman instability,[2]
- Cherenkov instability,[3]
- Chute instability
- Coalescence instability,[4]
- Collapse instability
- Counter-streaming instability
- Cyclotron instabilities, including:
  - Alfven cyclotron instability
  - Electron cyclotron instability
  - Electrostatic ion cyclotron Instability
  - Ion cyclotron instability
  - Magnetoacoustic cyclotron instability
  - Proton cyclotron instability
  - Nonresonant Beam-Type cyclotron instability
  - Relativistic ion cyclotron instability
  - Whistler cyclotron instability
- Diocotron instability,[5] (similar to the Kelvin-Helmholtz fluid instability).
- Disruptive instability (in tokamaks)
- Double emission instability
- Drift wave instability
- Edge-localized modes [6]
- Electrothermal instability
- Farley-Buneman instability,[7]
- Fan instability
- Filamentation instability
- Firehose instability (also called Hose instability)
- Flute instability
- Free electron maser instability
- Gyrotron instability
- Helical instability (helix instability)
- Helical kink instability
- Hose instability (also called Firehose instability)
- Interchange instability
- Ion beam instability
- Kink instability
- Lower hybrid (drift) instability (in the Critical ionization velocity mechani
- Magnetic drift instability
- Magnetorotational instability (in accretion disks)
- Magnetothermal instability (Laser-plasmas) [8]
- Modulation instability
- Non-Abelian instability (see also Chromo-Weibel instability)
- Chromo-Weibel instability
- Non-linear coalescence instability
- Oscillating two stream instability, see two stream instability
- Pair instability
- Parker instability (magnetic buoyancy instability)
- Peratt instability (stacked toroids)
- Pinch instability
- Sausage instability
- Slow Drift Instability
- Tearing mode instability
- Two-stream instability
- Weak beam instability
- Weibel instability
- z-pinch instability, also called Bennett pinch instability

MHD as a model of large-scale instabilities

- Ideal MHD is frequently used to describe large-scale instabilities characterized by low frequencies and long wavelengths.
- Ideal MHD is often a questionable model is weakly collisional plasmas, but describes well instabilities that grow rapidly.
- Such instabilities tap into macroscopic sources of free energy
  - Current density, plasma pressure, field-line bending, sheared flows
- MHD instabilities can lead to nonlinear explosive behavior.
- MHD instabilities can also produce turbulence widely seen in Nature, and the dynamo effect that can produce magnetic fields.
- MHD is a useful point of departure, even when it needs to be improved by including kinetic or non-ideal effects.
- In astrophysical objects, heat conduction and radiative cooling can also be destabilizing.

General strategy for studying plasma stability

- Start from an initial equilibrium, e.g.,

$$\frac{\mathbf{J}_0 \times \mathbf{B}_0}{c} = \nabla p_0 \tag{2}$$

- Linearize the equations of MHD and discard higher order terms
- Slightly perturb that equilibrium
- If there exists a growing perturbation, the system is *unstable*
- If no growing perturbation exists, the system is *stable*
- Use a combination of numerical simulations, experiments, and observations to study nonlinear dynamics

Linearizing the equations of ideal MHD

- Following the procedure for waves, we represent the relevant fields as the sum of equilibrium ('0') and perturbed ('1') components

$$
\begin{aligned}
\rho(\mathbf{r},t) &= \rho_0(\mathbf{r}) + \rho_1(\mathbf{r},t) && (3)\\
\mathbf{V}(\mathbf{r},t) &= \mathbf{V}_1(\mathbf{r}) && (4)\\
p(\mathbf{r},t) &= p_0(\mathbf{r}) + p_1(\mathbf{r},t) && (5)\\
\mathbf{B}(\mathbf{r},t) &= \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r},t) && (6)
\end{aligned}
$$

  where $\mathbf{V}_0 = 0$ for a static equilibrium
- Assume that the perturbed fields are much weaker than the equilibrium fields
- Use the convention that the perturbed fields vanish at $t = 0$

Linearizing the equations of ideal MHD

- To zeroeth order, a static equilibrium is given by

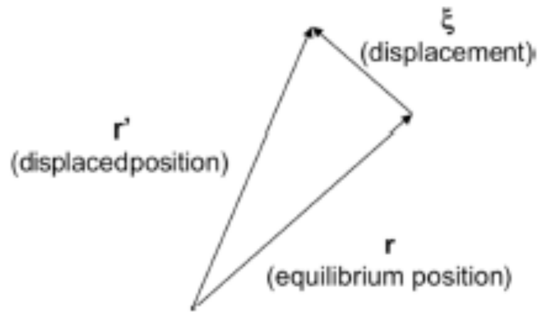
$$\nabla p_0 = \frac{(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0}{4\pi} \tag{7}$$

- Ignoring products of the perturbations gives

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\mathbf{V}_1 \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \mathbf{V}_1 && (8)\\
\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} &= -\nabla p_1 + \frac{(\nabla \times \mathbf{B}_0) \times \mathbf{B}_1}{4\pi} + \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi} && (9)\\
\frac{\partial \mathbf{B}_1}{\partial t} &= \frac{\nabla \times (\mathbf{V}_1 \times \mathbf{B}_0)}{c} && (10)\\
\frac{\partial p_1}{\partial t} &= -\mathbf{V}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{V}_1 && (11)
\end{aligned}$$

Note that $\mathbf{V}_1$ is the only time-dependent variable on the RHS of Eqs. 8, 10, and 11.

The displacement vector, ξ, describes how much the plasma is displaced from the equilibrium state



- If $\xi(\mathbf{r}, t = 0) = 0$, then the displacement is

$$\xi(\mathbf{r}, t) \equiv \int_0^t \mathbf{V}_1(\mathbf{r}, t')\mathrm{d}t' \tag{12}$$

- Its time derivative is just the perturbed velocity,

$$\frac{\partial \xi}{\partial t} = \mathbf{V}_1(\mathbf{r}, t) \tag{13}$$

Integrate the continuity equation with respect to time

- Put the linearized continuity equation in terms of ξ

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} &= -\mathbf{V}_1 \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \mathbf{V}_1 && (14)\\ &= -\frac{\partial \xi}{\partial t} \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \frac{\partial \xi}{\partial t} && (15)\end{aligned}$$

- Next we can integrate this

$$\int_0^t \frac{\partial \rho_1}{\partial t'} \mathrm{d}t' = \int_0^t \left[-\frac{\partial \xi}{\partial t'} \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \frac{\partial \xi}{\partial t'} \right] \mathrm{d}t' \quad (16)$$

which leads to a solution for ρ_1 in terms of just ξ

$$\rho_1(\mathbf{r}, t) = -\xi(\mathbf{r}, t) \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \xi(\mathbf{r}, t) \quad (17)$$

The linearized momentum equation in terms of ξ and $\mathbf{F}[\xi]$

- Using the solutions for ρ_1, $\mathbf{B}_1$, and p_1 we arrive at

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \mathbf{F}[\xi(\mathbf{r}, t)] \tag{21}$$

  which looks awfully similar to Newton's second law

- The ideal MHD force operator is

$$\begin{aligned} \mathbf{F}(\xi) &= \nabla(\xi \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \xi) \\ &+ \frac{1}{4\pi}(\nabla \times \mathbf{B}_0) \times [\nabla \times (\xi \times \mathbf{B}_0] \\ &+ \frac{1}{4\pi}\{[\nabla \times \nabla \times (\xi \times \mathbf{B}_0)] \times \mathbf{B}_0\} \end{aligned} \tag{22}$$

- The force operator is a function of ξ and the equilibrium, but not $\frac{\partial \xi}{\partial t}$

Finding the normal mode solution

- Separate the space and time dependences of the displacement:

$$\xi(\mathbf{r}, t) = \xi(\mathbf{r}) T(t) \tag{24}$$

- The linearized momentum equation becomes

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -\omega^2 T \tag{25}$$

$$-\omega^2 \rho_0 \xi(\mathbf{r}) = \mathbf{F}[\xi(\mathbf{r})] \tag{26}$$

so that $T(t) = e^{i\omega t}$ and the solution is of the form

$$\xi(\mathbf{r}, t) = \xi(\mathbf{r}) e^{i\omega t} \tag{27}$$

- Eq. 26 is an eigenvalue problem since **F** is linear
- The BCs determine the permitted values of ω^2
  - These can be a *discrete* or *continuous* set

The MHD force operator is *self-adjoint*

- The operator $\mathbf{F}(\boldsymbol{\xi})$ is *self-adjoint*. For any allowable displacement vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$

$$\int \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) \mathrm{d}\mathbf{r} = \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta}) \mathrm{d}\mathbf{r} \tag{29}$$

  For a proof, see Freidberg (1987)
- Self-adjointness is closely related to conservation of energy
  - If there is dissipation, $\mathbf{F}$ will not be self-adjoint

Finding the normal mode solution

- For a discrete set of ω^2, the normal mode solution is

$$\xi(\mathbf{r}, t) = \sum_n \xi_n(\mathbf{r}) e^{i\omega t} \tag{28}$$

  where ξ_n is the *normal mode* corresponding to its *normal frequency* ω_n
- Because $\mathbf{F}$ is self-adjoint, ω_n^2 must be real
- If $\omega_n^2 > 0 \; \forall n$, then the equilibrium is stable
- If $\omega_n^2 < 0$ for any n, then the equilibrium is unstable
- Stability boundaries occur when $\omega = 0$
- Now all we have to do is solve for a possibly infinite number of solutions!

The variational principle

- From these equations, we arrive at

$$\omega^2 = \frac{\delta W(\boldsymbol{\xi}, \boldsymbol{\xi})}{\delta K(\boldsymbol{\xi}, \boldsymbol{\xi})} \tag{39}$$

- Any $\boldsymbol{\xi}$ for which ω^2 is an extremum is an eigenfunction of $-\omega^2 \rho_0 \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})$ with eigenvalue ω^2
- If $\delta W < 0$, then there is an instability!

Strategy for the variational principle

- Choose a trial function

$$\xi = \sum_n a_n \phi_n \tag{40}$$

where ϕ_n are a suitable choice of basis functions subject to the normalization condition

$$K(\xi, \xi) = \text{const.} \tag{41}$$

- Minimize δW with respect to the coefficients a_n
- A lower bound for the growth rate γ is

$$\gamma \geq \sqrt{-\frac{\delta W}{K}} \tag{42}$$

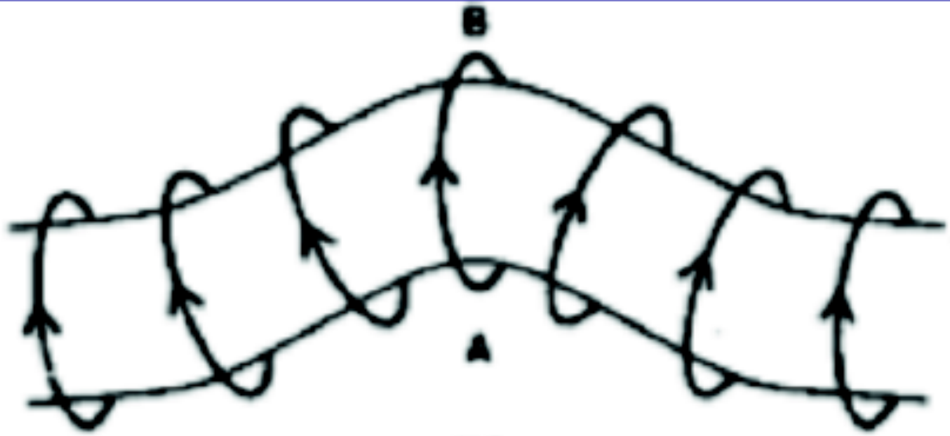
The intuitive form of energy principle

- After manipulation the energy principle can be written as

$$
\begin{aligned}
\delta W_P \;=\; & \frac{1}{2}\int \mathrm{d}\mathbf{r}\Big[\frac{|\mathbf{B}_{1\perp}|^2}{4\pi} + \frac{B^2}{4\pi}\left|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2 \\
& +\gamma p\left|\nabla\cdot\boldsymbol{\xi}\right|^2 - 2\left(\boldsymbol{\xi}_\perp\cdot\nabla p\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right) \\
& -J_\parallel\left(\boldsymbol{\xi}_\perp^*\times\mathbf{b}\right)\cdot\mathbf{B}_{1\perp}\Big] \qquad (48)
\end{aligned}
$$

- The first three terms are always stabilizing (in order):
  - The energy required to bend magnetic field lines (shear Alfvén wave)
  - Energy necessary to compress $\mathbf{B}$ (compr. Alfvén wave)
  - The energy required to compress the plasma (sound wave)
- The remaining two terms can be stabilizing or destabilizing:
  - Pressure-driven (interchange) instabilities (associated with $\mathbf{J}_\perp$)
  - Current-driven (kink) instabilities (associated with $\mathbf{J}_\parallel$)

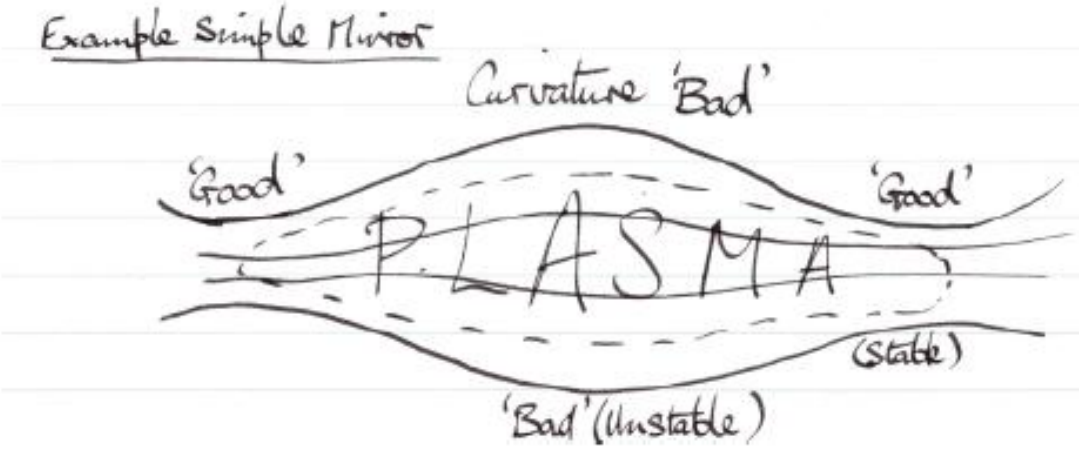
The kink instability



(b)

- Magnetic pressure is increased at point A where the perturbed field lines are closer together
- Magnetic pressure is decreased at point B where the perturbed field lines are separated
- A magnetic pressure differential causes the perturbation to grow
- The kink instability could be stabilized by magnetic field along the axis of the flux rope
  - Tension becomes a restoring force
- Usually long wavelengths are more unstable

Good curvature vs. bad curvature



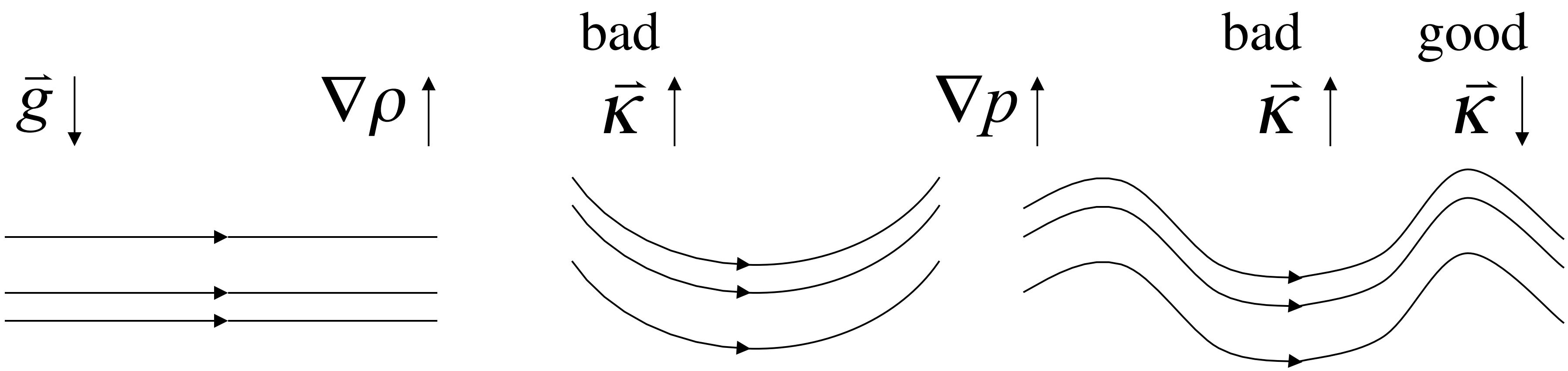
- Pressure-driven or interchange instabilities occur when

$$\boldsymbol{\kappa} \cdot \nabla p > 0 \tag{42}$$

where $\boldsymbol{\kappa} \equiv \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}$ is the curvature vector. This is a necessary but not sufficient criterion for pressure-driven instabilities.

- Instability occurs when it is energetically favorable for the magnetic field and plasma to switch places
- Usually short wavelengths are most unstable

Pressure-Gradient Driven MHD Instabilities



$\vec{k}\cdot\vec{B}=0$

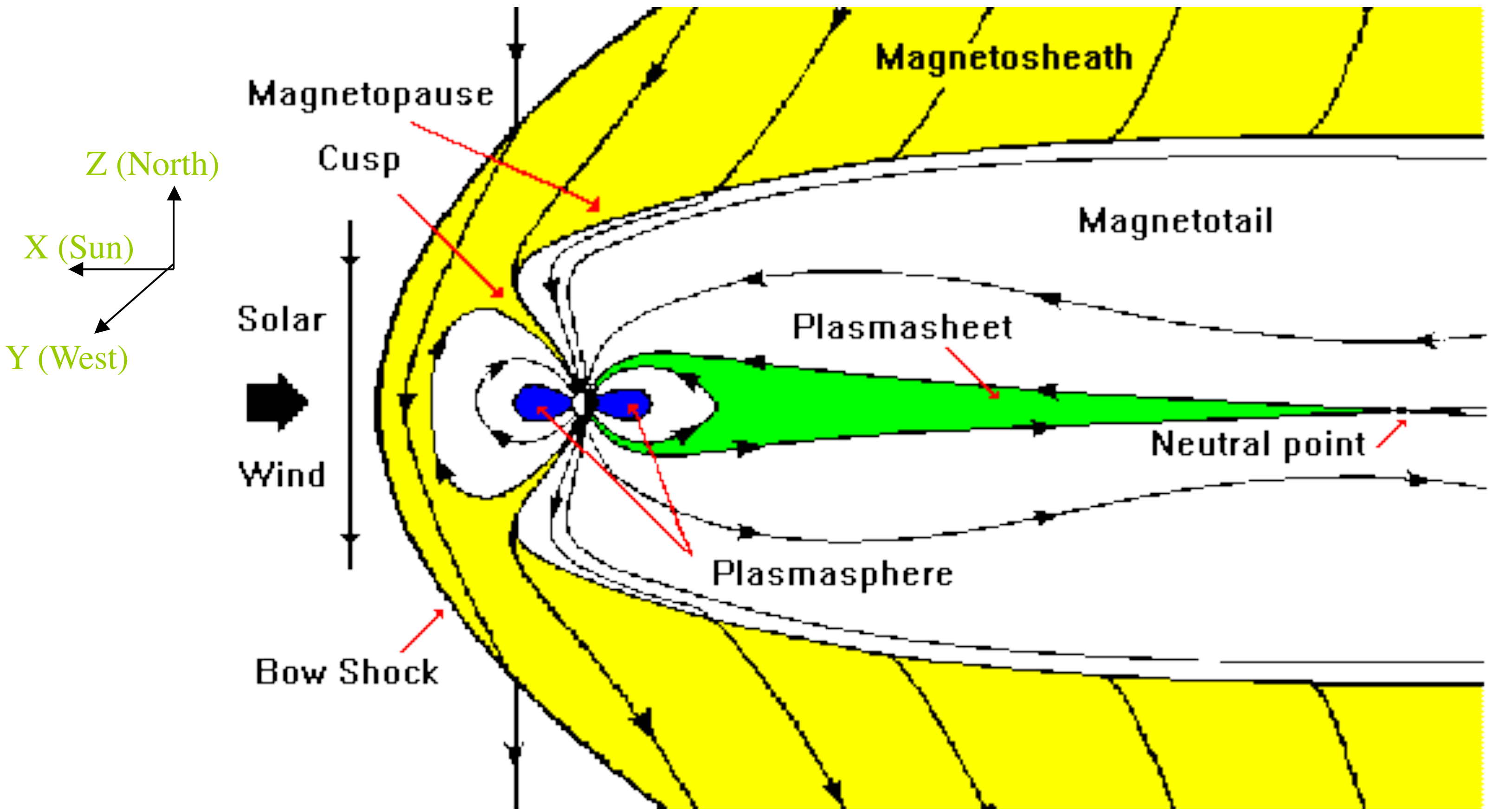
$\vec{k}\cdot\vec{B}\approx 0$

Rayleigh-Taylor -Parker

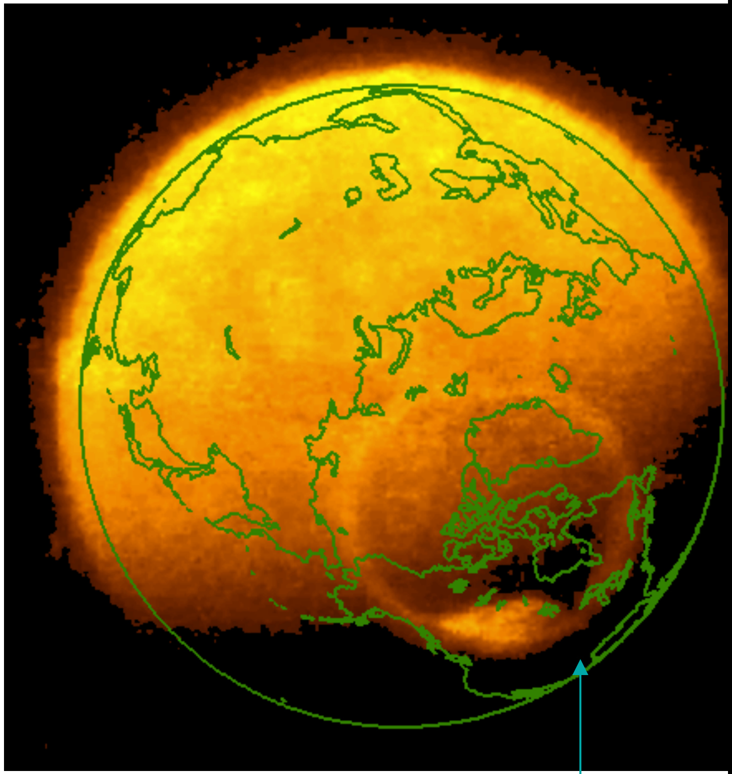
Interchange

Ballooning

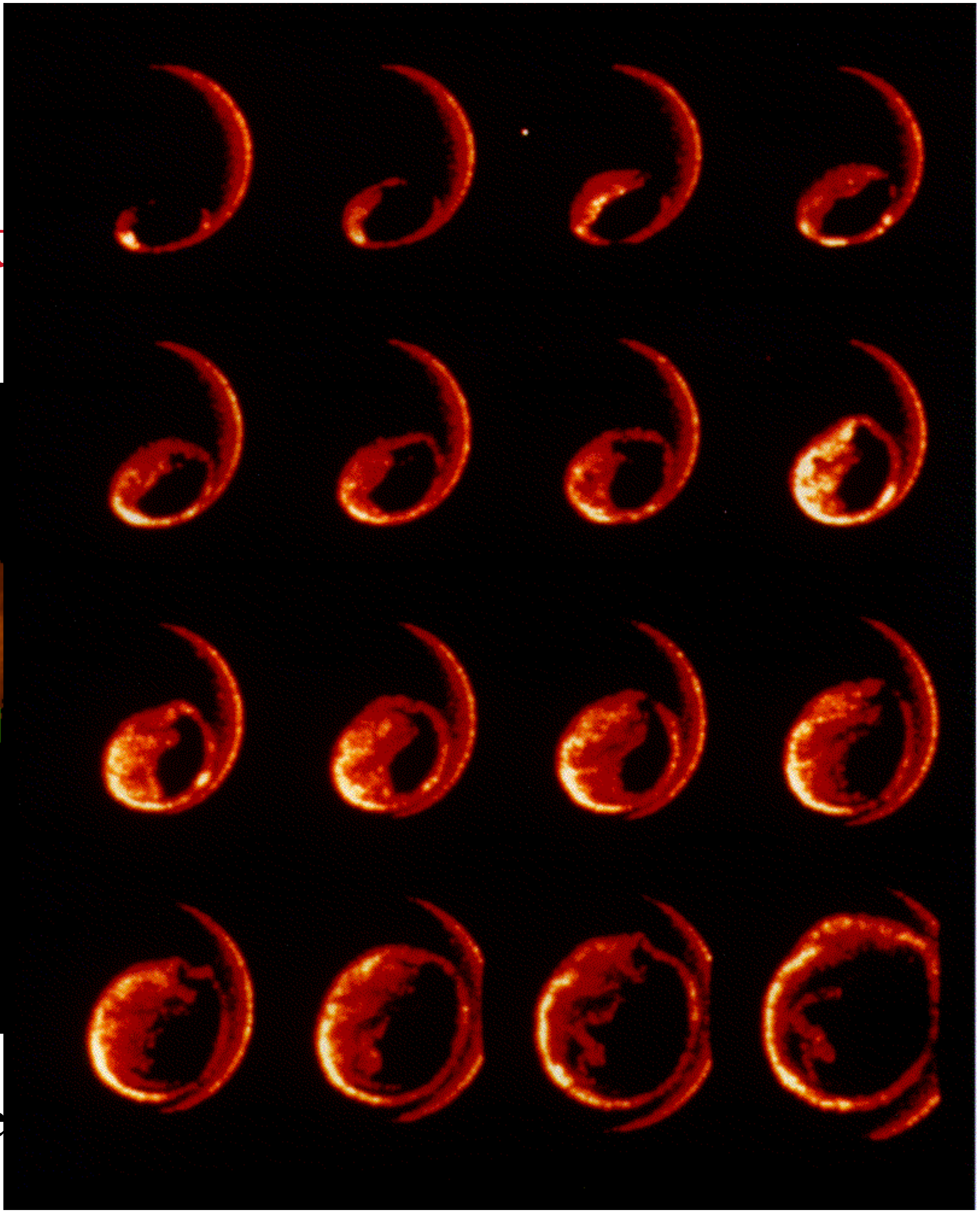
Substorm Onset:
Where does it occur?



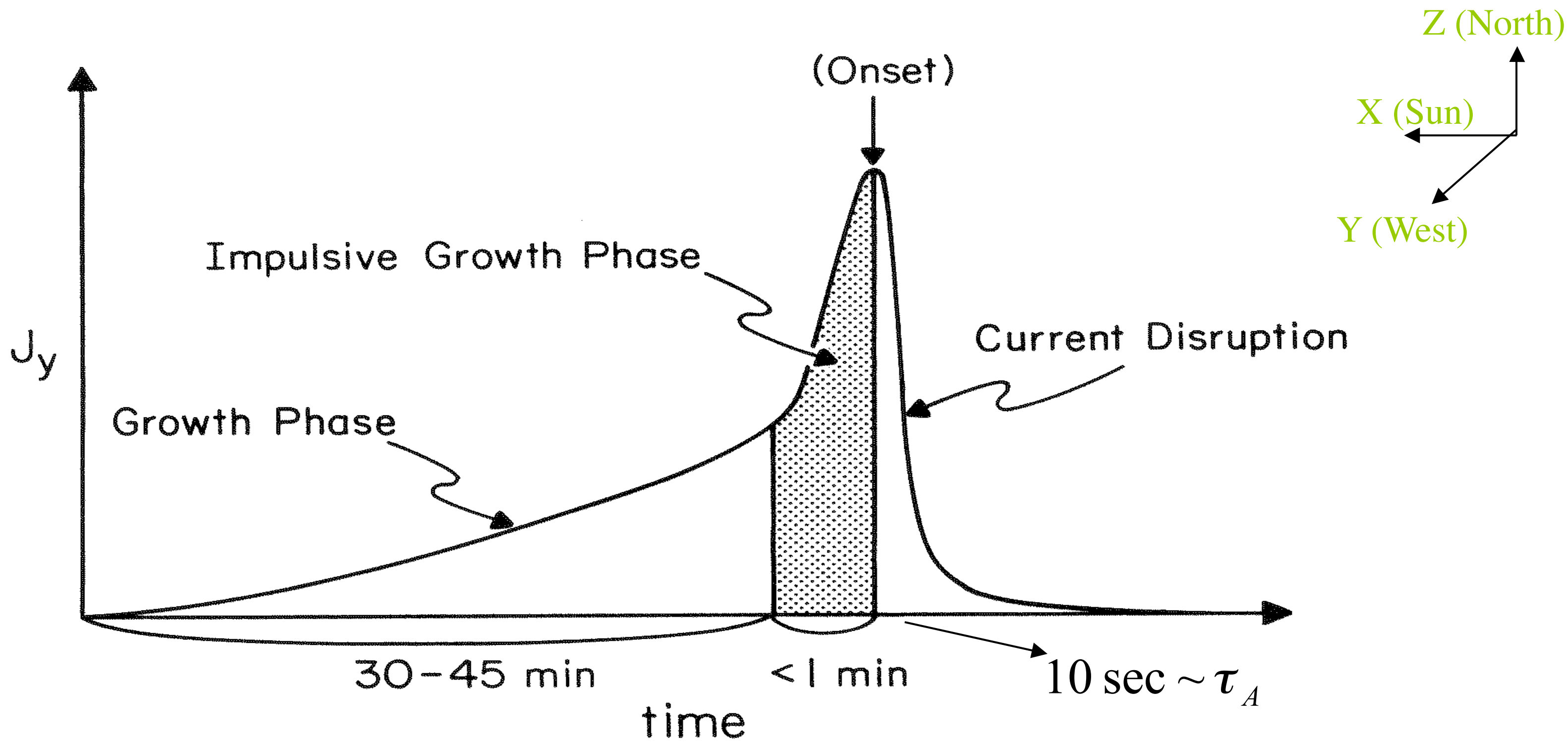
Substorm Onset



Auroral bulge



Substorm Onset:
When does it occur?

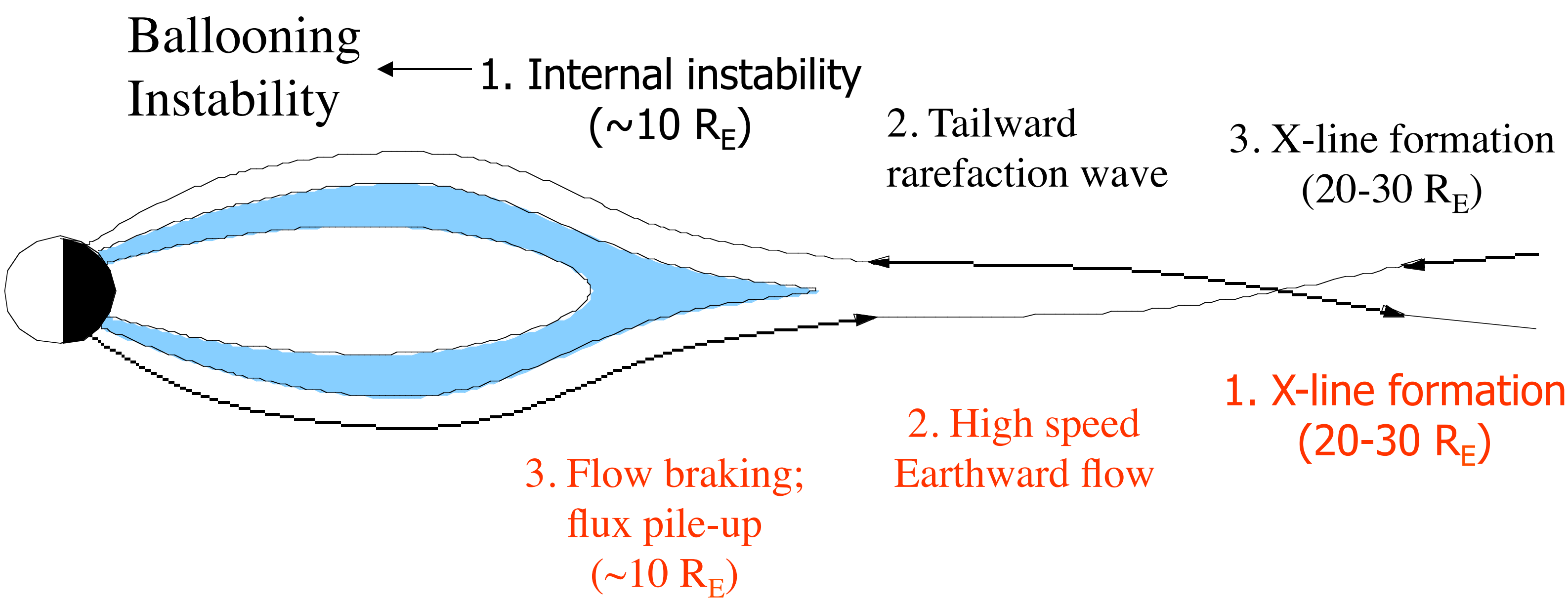


(Ohtani et al., 1992)

Growth Expansion Recovery

Substorm Onset:
How is it triggered?

Near-Tail Instability [e.g. A. Roux et al., 1991, Lui et al., 1992; Erickson et al., 2000]



Mid-Tail Reconnection [e.g. Shiokawa et al., 1998]

Substorm Onset:
The "2-minute" problem

"This year's substorm conference (ICS-4, 1998) was one of the most successful yet with over 250 in attendance. All existing paradigms were discussed at length and during the wrap-up session it was realized that ***only the time of the events within 2 minutes of onset were still seriously under debate***. Since it seems somewhat foolish for a couple of hundred scientists to travel half way around the world to argue over ***two minutes*** of geomagnetic activity, the substorm problem was declared solved and no more substorm conferences are being planned."

--- Y. Kidme (Kamide?)

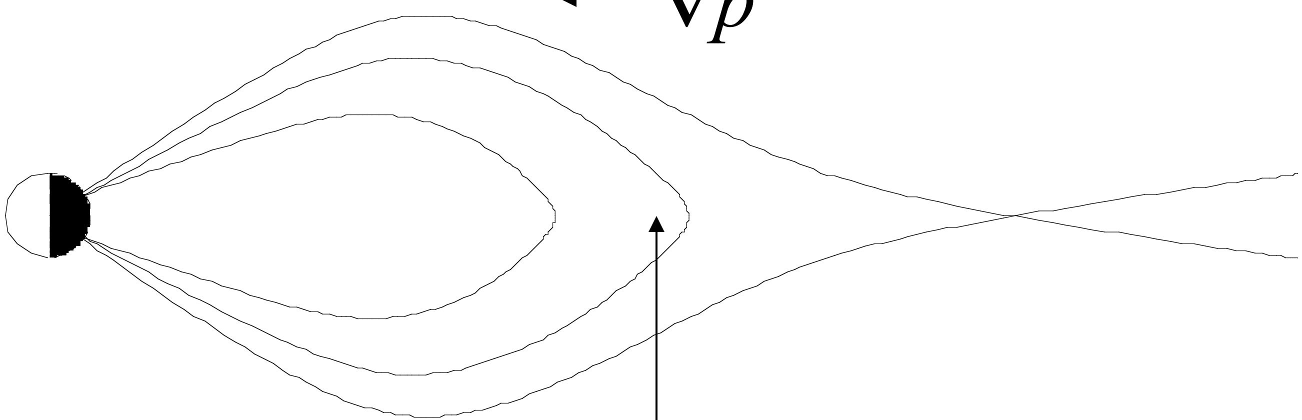
(AGU-SPA Newsletter, April 1, 1998)

Is Near-Earth Magnetotail Ballooning Unstable?

$\leftarrow \;\; \vec{\kappa}$

"bad curvature"

$\leftarrow \;\; \nabla p$

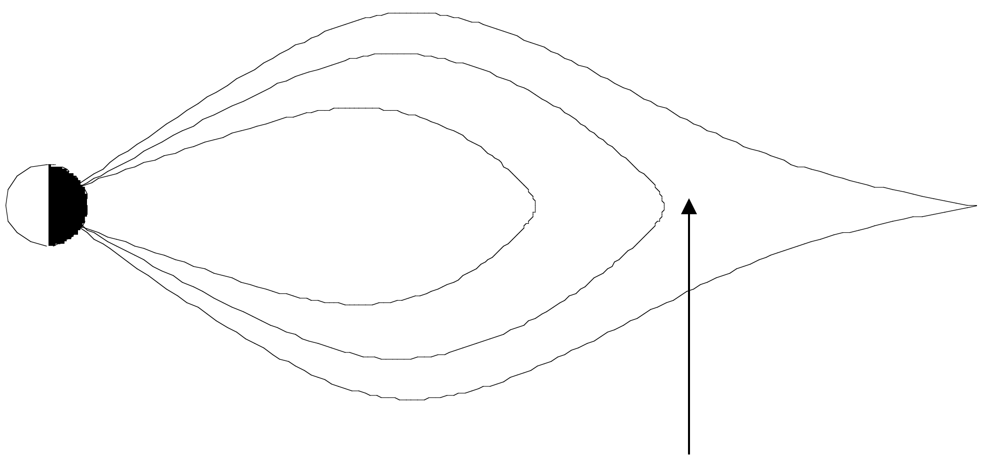


$\text{Near-Tail}\ (7 \sim 10\ R_E)$

$\beta_{eq} \sim O(1) - O(100)$!!

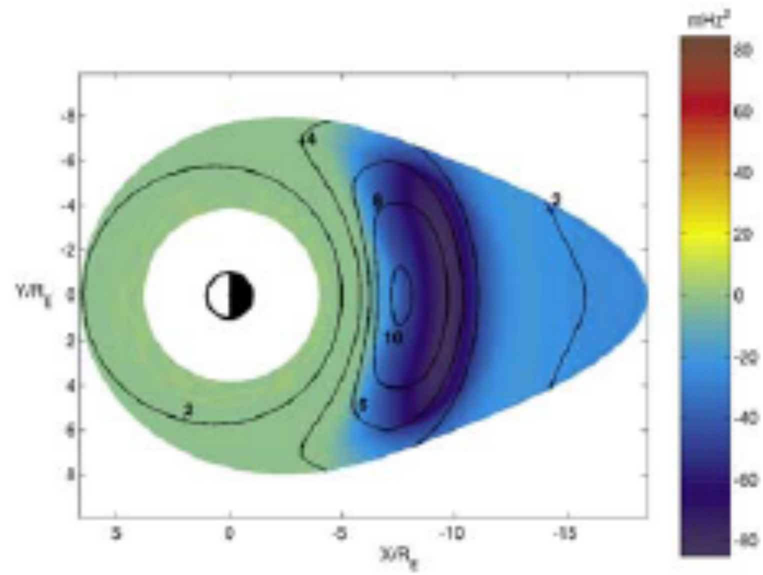
Is Near-Earth Magnetotail Ballooning Unstable? Yes …

But only when beta, field line length, gamma, CS width, ky is in the unstable parameter regime: near-tail, growth phase



Near - Tail $(7 \sim 10\ R_E)$

$\beta_{eq} \sim O(1) - O(100)$

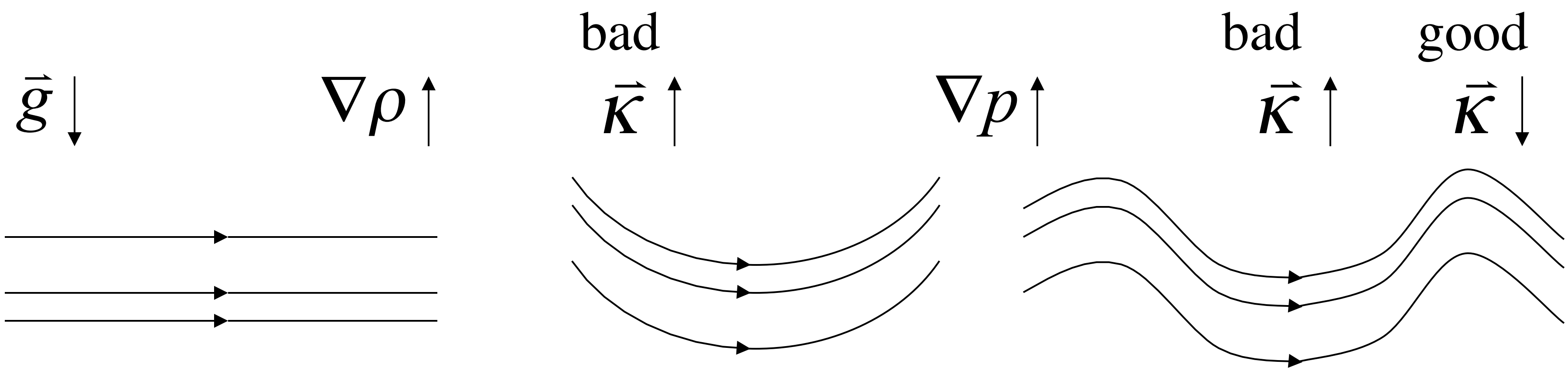


(Cheng and Zaharia, 2004)

Nonlinear Ballooning: Detonation or Saturation?

- Linear growth too weak – sub-Alfvenic
- Current sheet disruption is nonlinear
  - $\delta B \sim B_0$
  - Nonlinear spectrum of δB (Chen, Bhattacharjee, et al., 2003)
- Detonation model: explosive growth of nonlinear instability --- $(t_c\text{-}t)^{-\alpha}$ (Cowley and Artun, 1997)

Pressure-Gradient Driven MHD Instabilities



$\vec{k}\cdot\vec{B}=0$

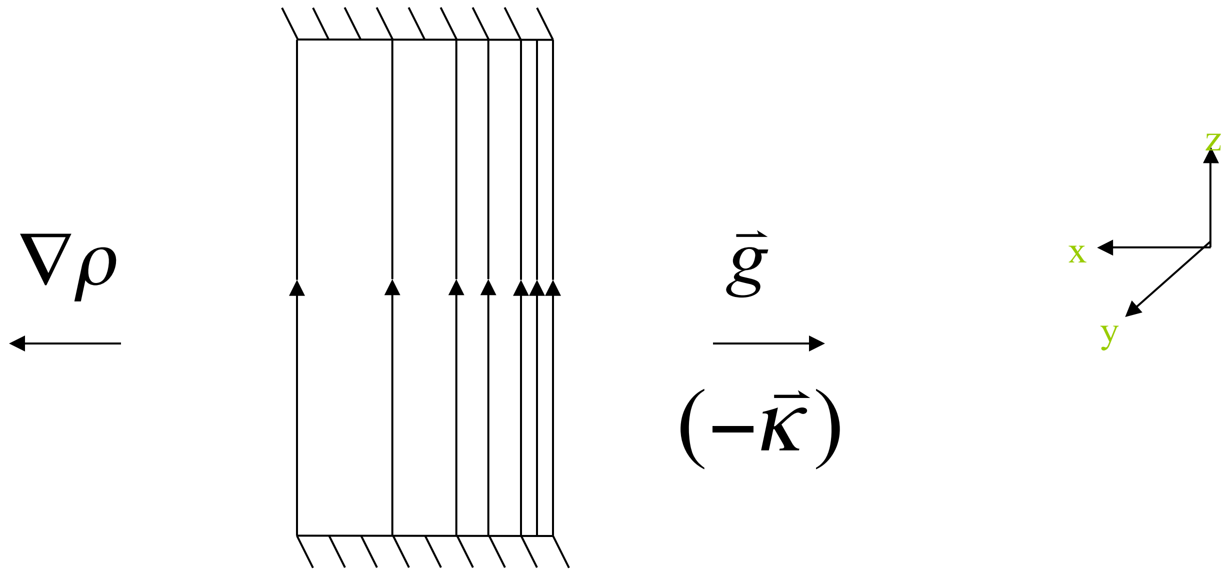
$\vec{k}\cdot\vec{B}\approx 0$

Rayleigh-Taylor
-Parker

Interchange

Ballooning

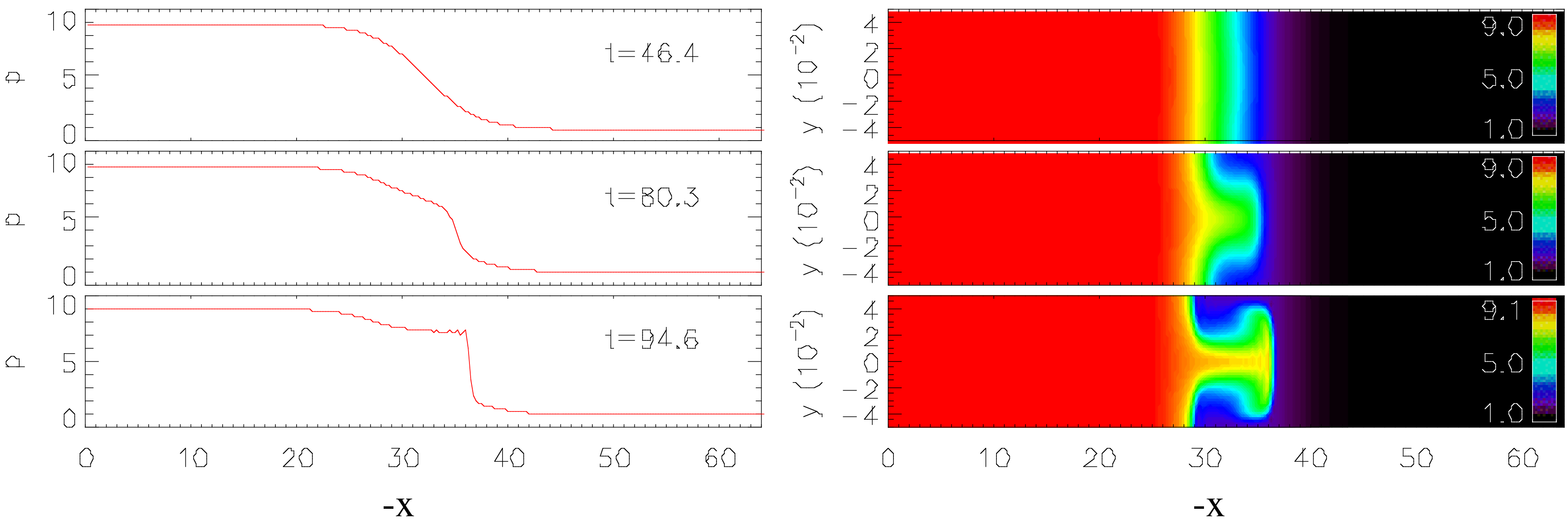
Nonlinear Rayleigh-Taylor-Parker (RTP) Instability



$$\rho = \rho(x), \quad p = p(x), \quad \vec{B} = B(x)\hat{z}, \quad \vec{g} = -g\hat{x}$$

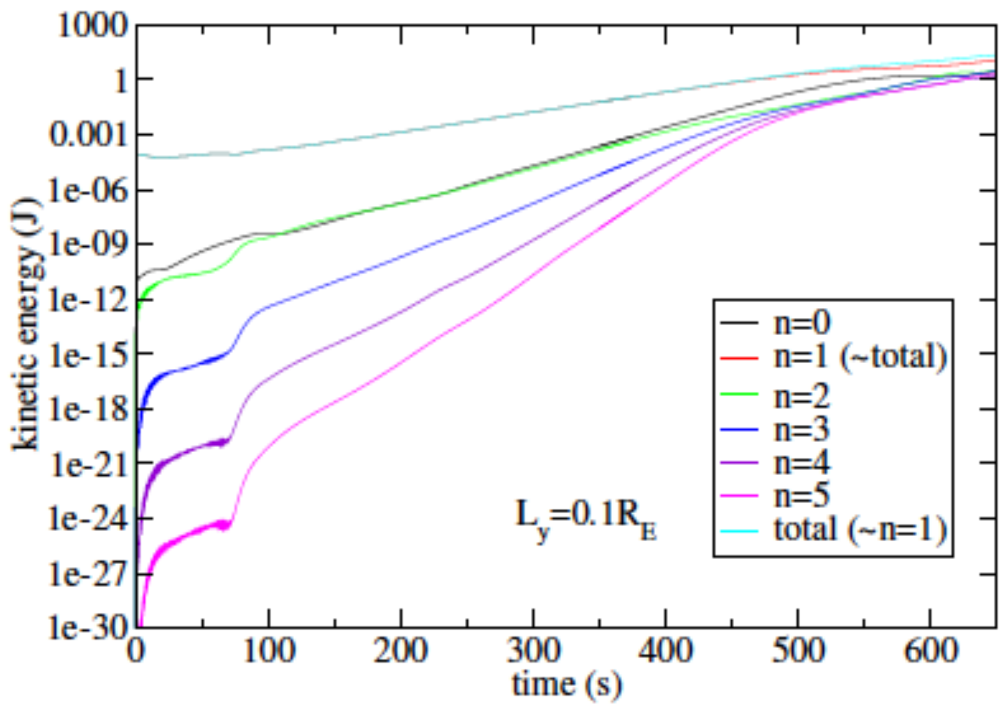
$$\frac{d\rho}{dx} > 0, \quad \frac{d}{dx}\left(p + \frac{B^2}{2}\right) = -\rho g$$

Nonlinear RTP: formation of contact discontinuity



(Zhu et al, 2005)

Nonlinear Ballooning Growth: Initial Spectrum: n=1



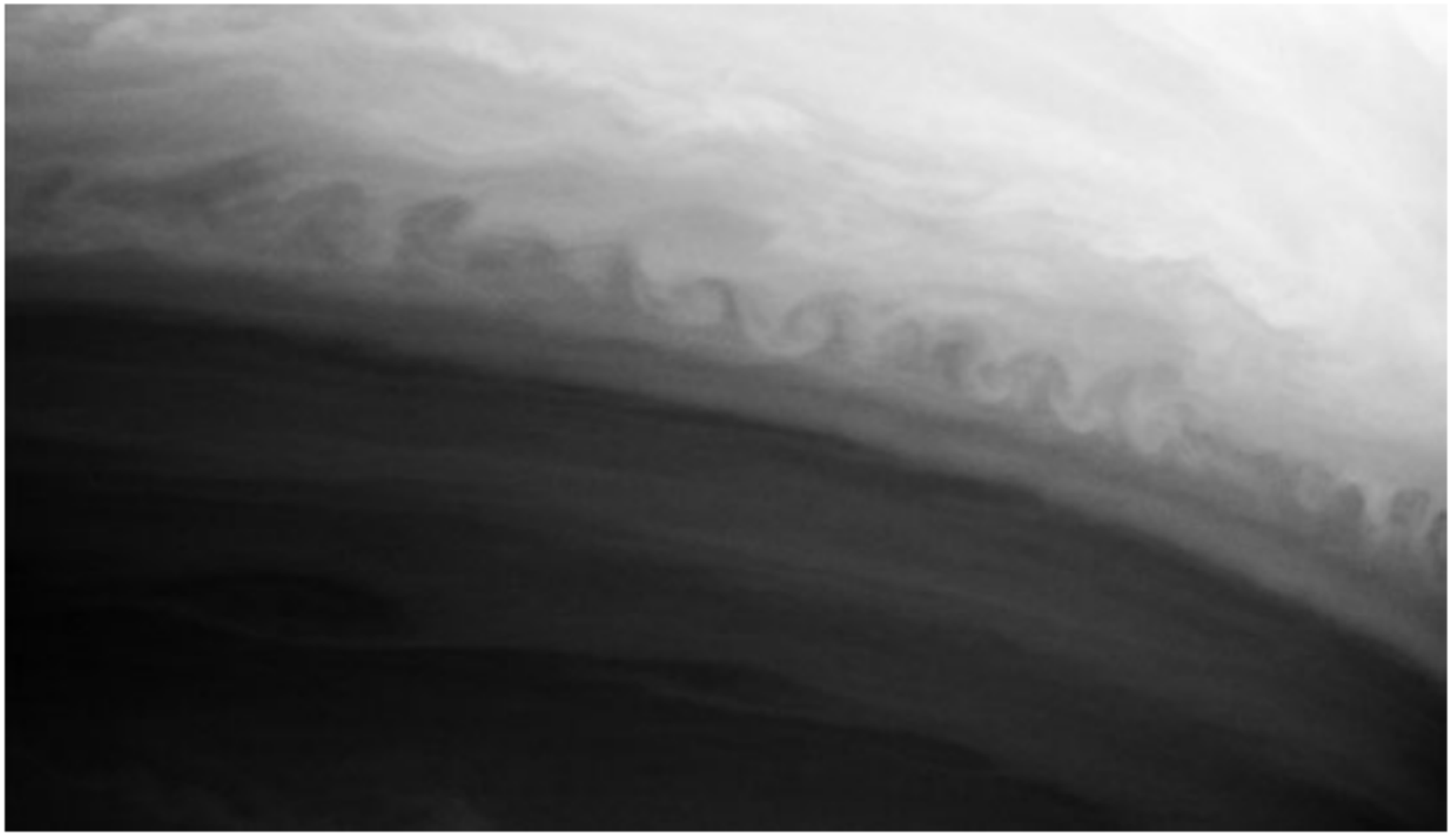
- $k_y = 2n\pi/L_y$;   n — Fourier mode number in y-direction.
- All higher-n Fourier modes are gradually excited. The growth of all modes slows down and saturates in nonlinear phase.

Overstable modes

- We said that there are no overstable modes in ideal MHD, but this changes in the presence of sheared flows
- This also occurs when we go beyond MHD and include, e.g.,
  - Radiative cooling
  - Anisotropic thermal conduction
- Examples include (e.g., Balbus & Reynolds 2010)
  - Magnetothermal instability
  - Heat flux buoyancy instability

  These are important in the intracluster medium of galaxy clusters.

The Kelvin-Helmholtz instability results from velocity shear

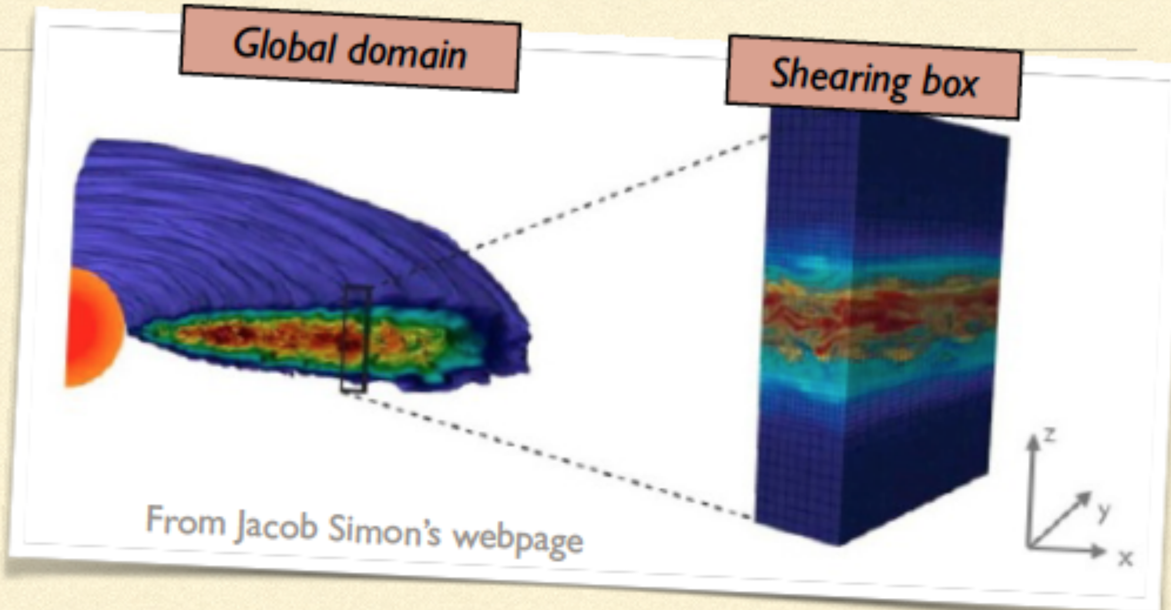


- Results in characteristic Kelvin-Helmholtz vortices
- Above: Kelvin-Helmholtz instability in Saturn's atmosphere

ACCRETION DISKS

- Due to angular momentum conservation, matter can rarely accrete directly into a central mass.
- Accretion is controlled by the dynamics and structure of disk object that forms.
- But observed accretion rates are **far too large** to be explained by the molecular viscosity.

- **Turbulence** is an obvious panacea.
  - Transport could be enhanced by orders of magnitude.
  - Hopefully independent of Reynolds numbers.



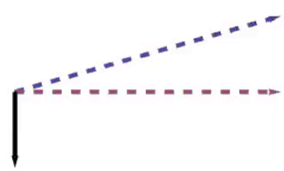
- **But what causes the turbulence?**
- All indications are the hydrodynamic disks are stable.
- Currently accepted theory (Balbus & Hawley 1991) relies on the disk being ionized:
  - The ***magnetorotational instability*** **(MRI)** causes the system to become turbulent, leading to strong outward angular momentum transport.

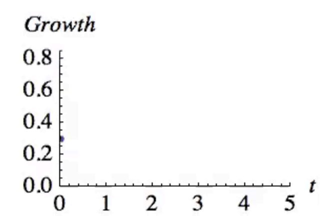
The Linear MRI involves a force operator that is non-self-adjoint

- Ideal MHD problems involving sheared plasma flows are non-self-adjoint (Frieman and Rotenberg, 1960)

$$\rho \frac{\partial^2 \xi}{\partial t^2} + 2\rho v.\nabla \frac{\partial \xi}{\partial t} - F\{\xi\} = 0$$

- Non-self-adjointness persists in the presence of dissipation.
- Eigenmodes are non-orthogonal.
- Non-orthogonality of eigenmodes allows for transient faster than the least unstable eigenmode.





- **Method:** choose a norm and maximize the solution at a chosen time.

A toy example (Trefethen and Embree 2005, Schmid 2007, Camporeale 2012)

Consider

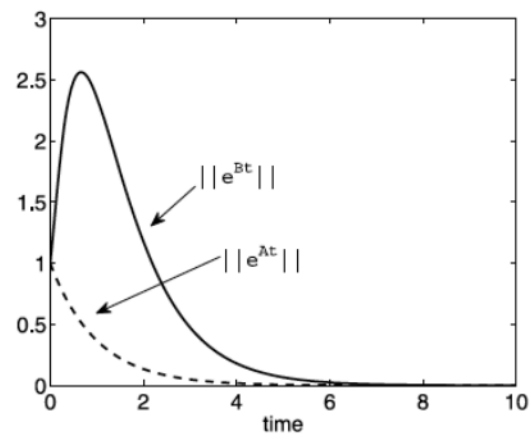
$$\frac{d\phi(t)}{dt} = A\phi(t),$$

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$$

A is normal but B is not. Both have the same eigenvalues. The growth function

$$G(t) = \frac{\|\phi(t)\|}{\|\phi(0)\|} = \frac{\|e^{\mathbf{A}t}\phi(0)\|}{\|\phi(0)\|},$$

Fig. 1 Time evolution of the norm of the exponential matrix for the matrices **A** and **B**. The *curves* shown bound by above the growth function $G(t)$ for any possible initial perturbation. The matrix **B**, being non-normal can support a transient growth



Ideal, resistive, and kinetic instabilities

- Ideal instabilities are usually the strongest and most unstable
  - Current-driven vs. pressure-driven
- Resistive instabilities are stable unless $\eta \neq 0$
  - Growth rate is usually slower than ideal instabilities
  - Often associated with magnetic reconnection
- Kinetic instabilities are often microinstabilities that occur when the distribution functions are far from Maxwellian
  - Two-stream, Weibel, Buneman, etc.
  - Important for near-Earth space plasmas, laboratory plasmas, cosmic ray interactions with ambient plasma, and dissipation of turbulence