Fluids and Plasmas, MHD, Instabilities, Waves and Turbulence

Marco Velli
Earth, Planetary and Space Sciences

- *Kinetic theory -> fluids -> plasmas*
- *MHD*
- *A fluid instability: why the solar wind is supersonic*
- *MHD waves, spherically polarized Alfvén waves, switchbacks and solar wind turbulence*
- *MHD invariants and Magnetic Reconnection as instability*

FLUID THEORIES

$$\frac{\partial f}{\partial t} + \vec{\nabla} \cdot (f\vec{v}) + \vec{\nabla}_{\vec{v}} \cdot (f\vec{a}) = 0$$

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{\vec{F}}{m} \cdot \vec{\nabla}_{\vec{v}} f = 0.$$

$$\vec{F} = \vec{F}_{lent.var.} + \vec{F}_{coll}$$

$$-\frac{\vec{F}_{coll}}{m} \cdot \vec{\nabla}_{\vec{v}} f = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\vec{\nabla}} f + \frac{\vec{F}}{m} \cdot \vec{\nabla}_{\vec{v}} f = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

MOMENTS

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\, \vec{u}) = 0$$

$$\rho \frac{du_i}{dt} = -\frac{\partial P_{ik}}{\partial x_k} + n\, F_i .$$

$$P_{ik} = P\, \delta_{ik} + \Pi_{ik} ,$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \tfrac{3}{2}P\right) + \frac{\partial}{\partial x_k}(\tfrac{1}{2}\rho u^2 + \tfrac{3}{2}P)u_k + u_i\, P_{ik} + q_k]$$

$$\vec{q} = n\langle \vec{w}(\tfrac{1}{2}mw^2)\rangle$$

$$-n\, F_i u_i = \left(\frac{\partial}{\partial t}(n\langle \tfrac{1}{2}mv^2\rangle)\right)_{coll},$$

$$\tfrac{3}{2}\rho^{5/3}\, \frac{d}{dt}\left(P\, \rho^{-5/3}\right) = -\Pi_{ik} \frac{\partial u_i}{\partial x_k} - \frac{\partial q_k}{\partial x_k} .$$

When does an ionized gas become a plasma ?

$$\lambda_D = \left(\frac{kT}{4\pi n e^2}\right)^{1/2} \qquad N_D = \frac{4\pi}{3} n \lambda_D^3 \gg 1 \qquad \omega_{pe}^2 = \frac{4\pi n_e e^2}{m_e}$$

Saha equation: the ionization of a hydrogen gas is essentially complete at temperatures of order 10^4 K, much less than the temperature corresponding to the ionization potential

$$T \simeq I/k \simeq 1.58 \times 10^5 \, K$$

Also important is that a plasma reaches 90% conductivity with an ionization degree of only 8%.

Collisions in a plasma

$$\frac{Z_1 Z_2 e^2}{b} \simeq \frac{3}{2} kT,$$

The collision cross-section will therefore be given by:

$$\sigma_c = \pi b^2 = \frac{4\pi Z_1^2 Z_2^2 e^4}{(3kT)^2},$$

and the collision frequency:

$$\nu_c = n \sigma_c v_T = \frac{4\pi Z_1^2 Z_2^2 e^4 n}{m^{1/2}(3kT)^{3/2}},$$

Thermal Conductivity

$$\kappa \sim v_T^2 / \nu_c \sim T^{5/2}$$

Collisions are also the way through which a plasma thermalizes, i.e. through which a plasma containing particle populations with different temperatures reaches thermal equilibrium. In a collision, energy may be transferred from the particle of higher energy to the lower energy one. Consider then the case of populations of electrons and ions both out of thermodynamic equilibrium but with comparable energies. It may be shown that collisions lead to equilibrium among particles of the same species on a different timescale compared to that required for thermal equilibrium across species. For collision between identical particles, the characteristic timescale τ required to reach equilibrium is given by

$$\tau_{ee} \simeq (\nu_{ee})^{-1} \simeq \left(\frac{m_e}{m_i}\right)^{1/2} \tau_{ii}\,.$$

Thermal equilibrium across species implies collisions between electrons and ions: in one collision an electron can only lose a fraction of order (m_e/m_i) of its energy. Reaching thermal equilibrium therefore requires a time

$$\tau_{ei} \simeq \left(\frac{m_i}{m_e}\right)^{1/2} \tau_{ii} \simeq \left(\frac{m_i}{m_e}\right) \tau_{ee}\,.$$

1 Fluid Closure: MHD equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U}) = 0$$

$$\rho \frac{\mathrm{D}\vec{U}}{\mathrm{D}t} = -\vec{\nabla} P + \frac{1}{c} \vec{J} \times \vec{B} = -\vec{\nabla} P + \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \times \vec{B}$$

$$\vec{J} = \frac{c}{4\pi} (\vec{\nabla} \times \vec{B}) \quad \vec{E} = -\frac{1}{c} \vec{U} \times \vec{B} + \frac{\vec{J}}{\sigma}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{U} \times \vec{B} + \vec{\nabla} \times \frac{\eta c^2}{4\pi} \nabla \times \vec{B} \qquad \eta_m = \frac{\eta c^2}{4\pi}$$

$$\frac{1}{\gamma - 1} \rho^{\gamma} \frac{\mathrm{D}}{\mathrm{D}t} \left(P \rho^{-\gamma} \right) = \frac{4\pi}{c^2} \eta J^2$$

$$q = \frac{1}{4\pi} (\vec{\nabla} \cdot \vec{E})$$

$$\rho\frac{\mathrm{D}U_i}{\mathrm{D}t} = -\frac{\partial}{\partial x_i}(P + \frac{B^2}{8\pi}) + \frac{1}{4\pi}B_k\frac{\partial B_i}{\partial x_k} = -\frac{\partial}{\partial x_i}(P + \frac{B^2}{8\pi}) + \frac{1}{4\pi}\frac{\partial(B_iB_k)}{\partial x_k}$$

$$\rho\frac{\mathrm{D}U_i}{\mathrm{D}t} = \frac{\partial}{\partial x_k}\mathrm{T_{ik}} \qquad \mathrm{T_{ik}} = -(P + \frac{B^2}{8\pi})\delta_{ik} + \frac{1}{4\pi}B_iB_k$$

The relative role of the magnetic field associated forces and pressure forces is given by the plasma beta

$$\beta = \frac{P}{B^2/8\pi} = \frac{8\pi P}{B^2}$$

The MHD equations may be written in conservative form as

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_k}\left[\rho U_iU_k + (P + \frac{B^2}{8\pi})\delta_{ik} - \frac{1}{4\pi}B_iB_k\right] = 0.$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho U^2 + \frac{P}{\gamma - 1} + \frac{B^2}{8\pi}\right) + \frac{\partial}{\partial x_i}\left[U_i(\tfrac{1}{2}\rho U^2 + \frac{\gamma P}{\gamma - 1}) + \frac{c}{4\pi}(\vec{E}\times\vec{B})_i\right] = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{U}) = 0 \qquad \frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_k}(U_kB_i - U_iB_k) = \eta_m\nabla^2\vec{B} - \vec{\nabla}\eta_m\times(\vec{\nabla}\times\vec{B})$$

Virial theorem and plasma equilibria

Force tensor

$$G_{ik} = P_{ik} + \frac{B^2}{8\pi}\delta_{ik} - \frac{B_iB_k}{4\pi} = (P + \frac{B^2}{8\pi})\delta_{ik} + \Pi_{ik} - \frac{B_iB_k}{4\pi}.$$

Multiply by the ith component of x_i and integrate over the volume.

$$\int_V x_i(\frac{\partial}{\partial x_k}G_{ik})DV = 0$$

Integrating by parts you obtain

$$\int_S x_i G_{ik}\, dS_k - \int_V \frac{\partial x_i}{\partial x_k} G_{ik}\, dV = 0$$

Surface term can be made to vanish BUT

$$\mathbf{Tr}\,\mathrm{G} = 3\mathrm{P} + \frac{\mathrm{B}^2}{8\pi} \neq 0.$$

A plasma can not be confined by only the self-consistent EM field.
Adding Gravity and kinetic energy one gets

$$U = \int_V \frac{p}{\gamma - 1}\,\mathrm{dV},\; M = \int_V \frac{B^2}{8\pi}\,\mathrm{dV},$$

$$T = \int_V \frac{1}{2}\rho u^2\,\mathrm{dV},\; \phi = -\frac{1}{2}\int_V \int_{V'} \frac{G\rho\rho'}{|\vec{r} - \vec{r}'|}\,\mathrm{dV},\mathrm{dV}'$$

$$2T + 3(\gamma - 1)U + M + \phi + S = 0$$

The surface term is given by

$$S = -\int_S \left((p + \frac{B^2}{8\pi})\vec{r} - (\vec{r}\cdot\vec{B})\frac{\vec{B}}{4\pi}\right)\cdot\mathrm{d}\vec{S}$$

DYNAMICS OF THE INTERPLANETARY GAS AND MAGNETIC FIELDS*

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Received January 2, 1958

ABSTRACT

We consider the dynamical consequences of Biermann's suggestion that gas is often streaming outward in all directions from the sun with velocities of the order of 500–1500 km/sec. These velocities of 500 km/sec and more and the interplanetary densities of 500 ions/cm^3 (10^{14} gm/sec mass loss from the sun) follow from the hydrodynamic equations for a 3×10^6 ° K solar corona. It is suggested that the outward-streaming gas draws out the lines of force of the solar magnetic fields so that near the sun the field is very nearly in a radial direction. Plasma instabilities are expected to result in the thick shell of disordered field (10^{-5} gauss) inclosing the inner solar system, whose presence has already been inferred from cosmic-ray observations.

I. INTRODUCTION

Biermann (1951, 1952, 1957*a*) has pointed out that the observed motions of comet tails would seem to require gas streaming outward from the sun. He suggests that gas is often flowing radially outward in all directions from the sun with velocities ranging from 500 to 1500 km/sec; there is no indication that the gas ever has any inward motion. Biermann infers densities at the orbit of earth ranging from 500 hydrogen atoms/cm^3 on magnetically quiet days to perhaps 10^5/cm^3 during geomagnetic storms (Unsöld and Chapman 1949). The mass loss to the sun is 10^{14}–10^{15} gm/sec. It is the purpose of this paper to explore some of the grosser dynamic consequences of Biermann's conclusions.

For instance, we should like to understand what mechanism at the sun might conceivably be responsible for blowing away the required 10^{14}–10^{15} gm of hydrogen each second, with velocities of the order of 1000 km/sec. All known mechanisms, such as

How does a hot corona expand?

$$g_s = \frac{GM_\odot}{R_s^2} \quad r = \frac{R}{R_s} \quad \frac{\partial p}{\partial r} = -\frac{2m_p n g R_s}{r^2} \quad p = 2nkT$$

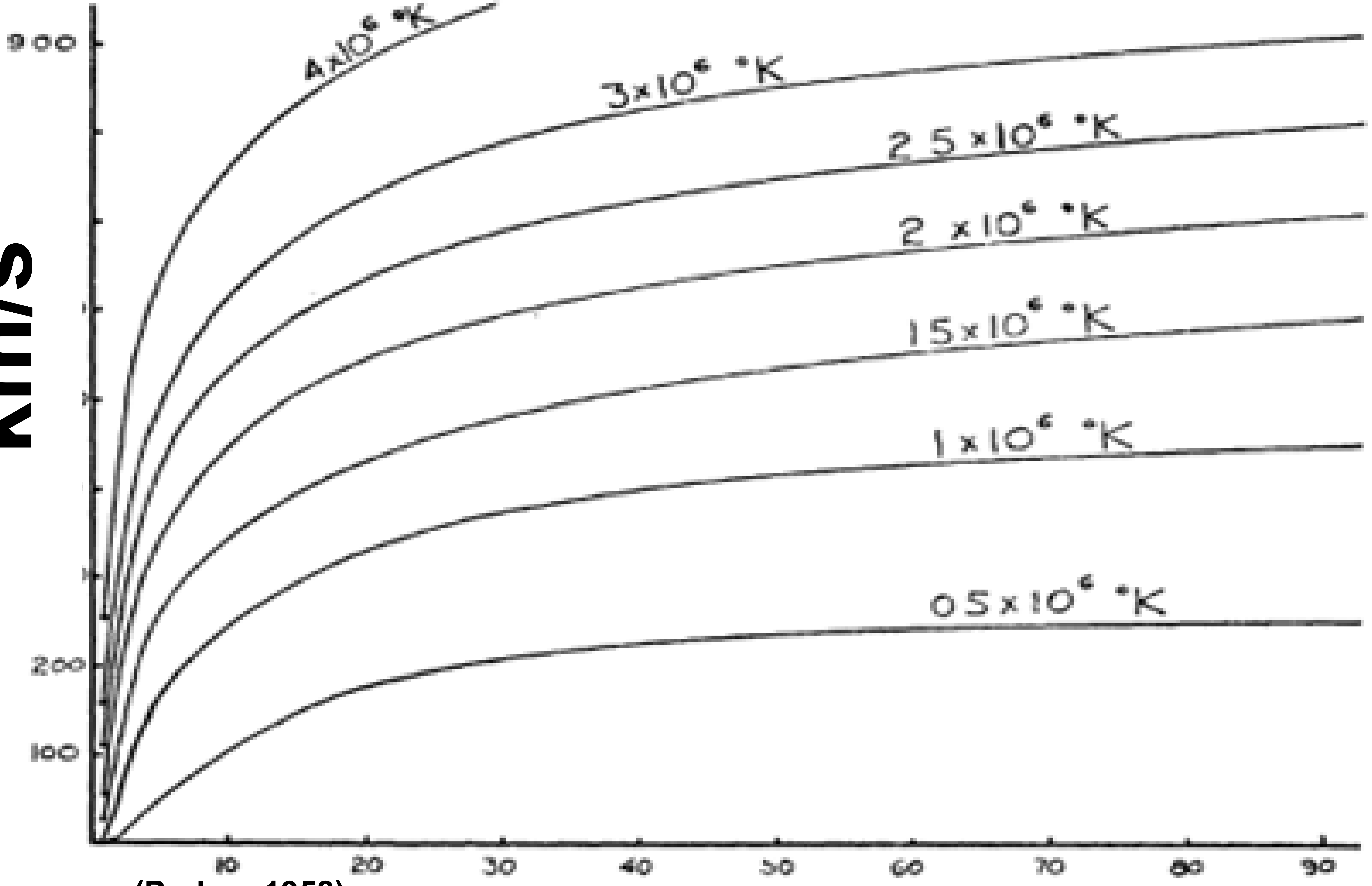
$$p(r) = p_0 \exp\left(-\int_1^r dr' \frac{m_p g_s R_s}{kT\, r^2}\right)$$

$T(r)$ falls slower than $1/r$ a finite pressure at infinity is required confine atmosphere. In a hot plasma atmosphere thermal conduction is proportional $k \sim T^{5/2}$ and therefore $T(r) \sim r^{-2/7}$

Since we know of no general pressure at infinity which could balance the $p(\infty)$ computed from equation (9) with the expected values of n, we conclude that probably it is not possible for the solar corona, or, indeed, perhaps the atmosphere of any star, to be in complete hydrostatic equilibrium out to large distances. We expect always to find some continued outward hydrodynamic expansion of gas—without considering the evaporation from the high-velocity tail of the Maxwellian distribution (Spitzer 1947; van de Hulst 1953).

Parker, 1958

wind sped
km/s



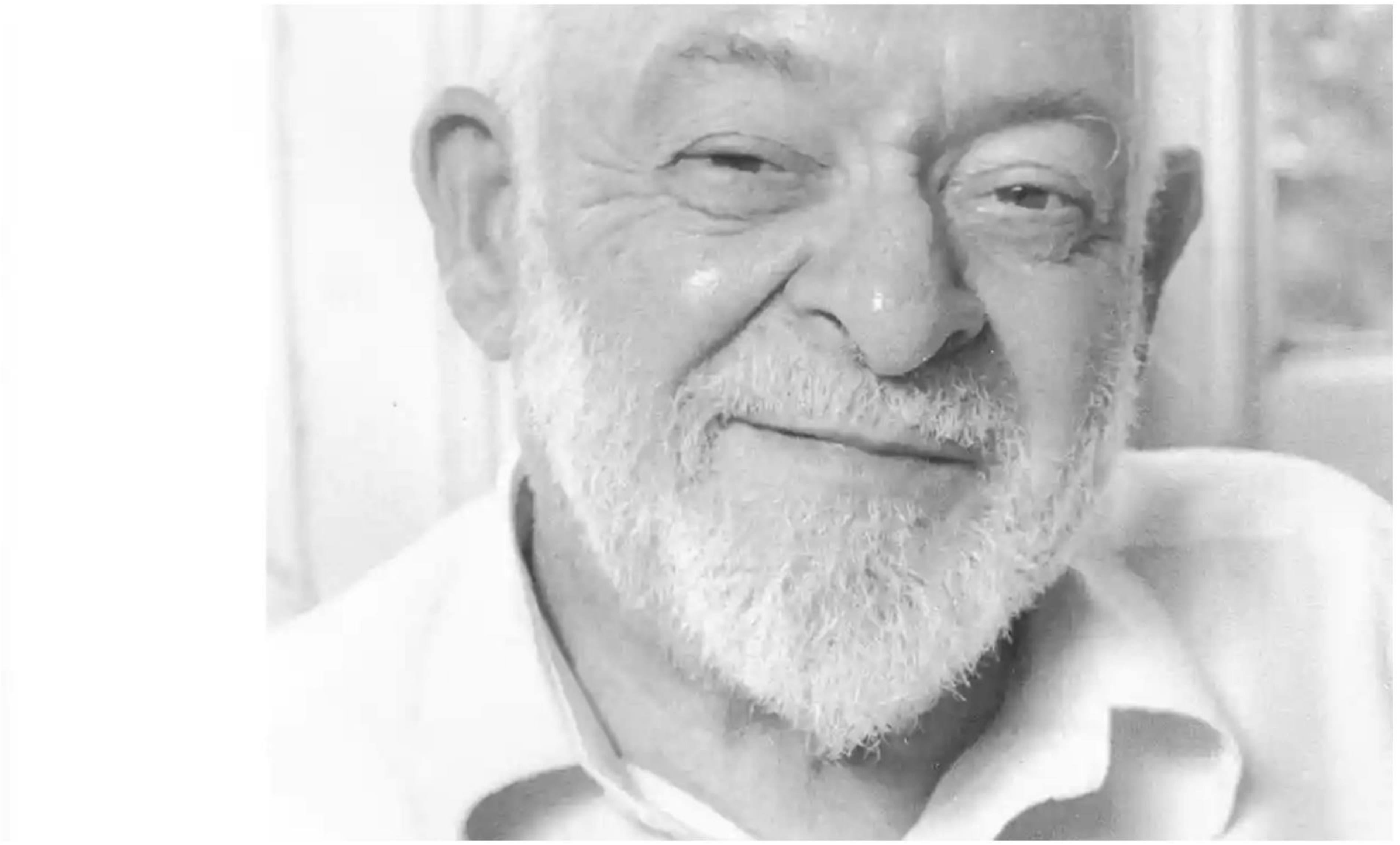
(Parker, 1958)

Distance to the Sun

Mestel, quoted in Roberts and Soward (1972): "were the temperature at the base of the solar corona 10^5 K rather than the generally accepted 10^6 K the total pressure far from the sun would suffice to suppress the solar wind entirely"

P_{ism} = 1.24 10^{-12} dyne/cm^2 confines a 10^5 K corona

Astronomer and astrophysicist who inspired generations of students and discovered the cooling law for white dwarf stars



Leon Mestel found that white dwarf stars – dead stars that are the endpoint of evolution of stars such as the sun – cooled over billions of years. Photograph: Cath Forrest

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Stellar winds and breezes

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(*Communicated by K. Stewartson F.R.S. – Received 13 September 1971 – Revised 10 January 1972*)

Steady stellar winds are generally divided into two classes: (i) the winds proper, for which the energy flux per unit solid angle, E_∞, is non-zero, and (ii) the breezes, for which $E_\infty = 0$.

The breezes may be distinguished from one another by the value of the ratio, g, of kinetic to thermal energy of the particles in the limit of large distance, r, from the stellar centre, or more precisely by

$$g = \lim_{r\to\infty} \frac{mv^2}{kT},$$

where $v(r)$ is breeze velocity, $T(r)$ is temperature, m is mean particle mass, and k is the Boltzmann constant. Solutions have previously been obtained for values of g in the range $0 \leqslant g \leqslant 1$, in which the breezes are subsonic everywhere with respect to the isothermal speed of sound. It is demonstrated here that two distinct solutions exist as $g \to \frac{5}{3}$, namely (in an obvious notation) the $g = \frac{5}{3}-$ and the $g = \frac{5}{3}+$ possibilities. It is shown that, if $g > \frac{5}{3}$ $(g < \frac{5}{3})$ the solutions are everywhere supersonic (subsonic) with respect to the adiabatic speed of sound. If $1 < g < \frac{5}{3}$, they possess a critical point, at which the isothermal speed of sound and the flow speed coincide.

The winds are examined in the limit $E_\infty \to 0$, and the relation with the breezes is studied. In particular, it is shown that, for $r \leqslant O(E_\infty^{-2/5})$, the winds satisfy the stellar breeze equations to leading order, and possess a critical point at $r = O(1)$. For $r > O(E_\infty^{-2/5})$, the solutions do not obey the breeze equations. They ultimately follow the Durney asymptotic law $[T = O(r^{-4/3})$, for $r \to \infty]$ for the winds. This demonstration of how the winds merge continuously into the breezes as $E_\infty \to 0$ is new.

The question of how the particle density (N_0) and temperature (T_0) at the base of the stellar corona determine the type of solution realized outside the star is examined. Even when the flow speed, v_0, at the base of the corona is subsonic, non-uniqueness can occur. In one domain of the (N_0, T_0) plane, two distinct types of breeze are possible; in another these, together with a wind $(E_\infty \neq 0)$, are permissible. Elsewhere (large N_0, moderate T_0) only a unique breeze exists or (small N_0 and/or T_0) a unique wind. In some domains (large T_0) no steady solution exists, unless the requirement that the corona is subsonic is relaxed. In this case, however, the problems of non-uniqueness are severely aggravated.

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‡ The National Center for Atmospheric Research is sponsored by the National Science Foundation.
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Stationary, spherically symmetric flows, isothermal approximation, sound speed c, $\gamma = 1$ ($r = R/R_s$)

$$\rho U r^2 = F_m \qquad p = c^2 \rho$$

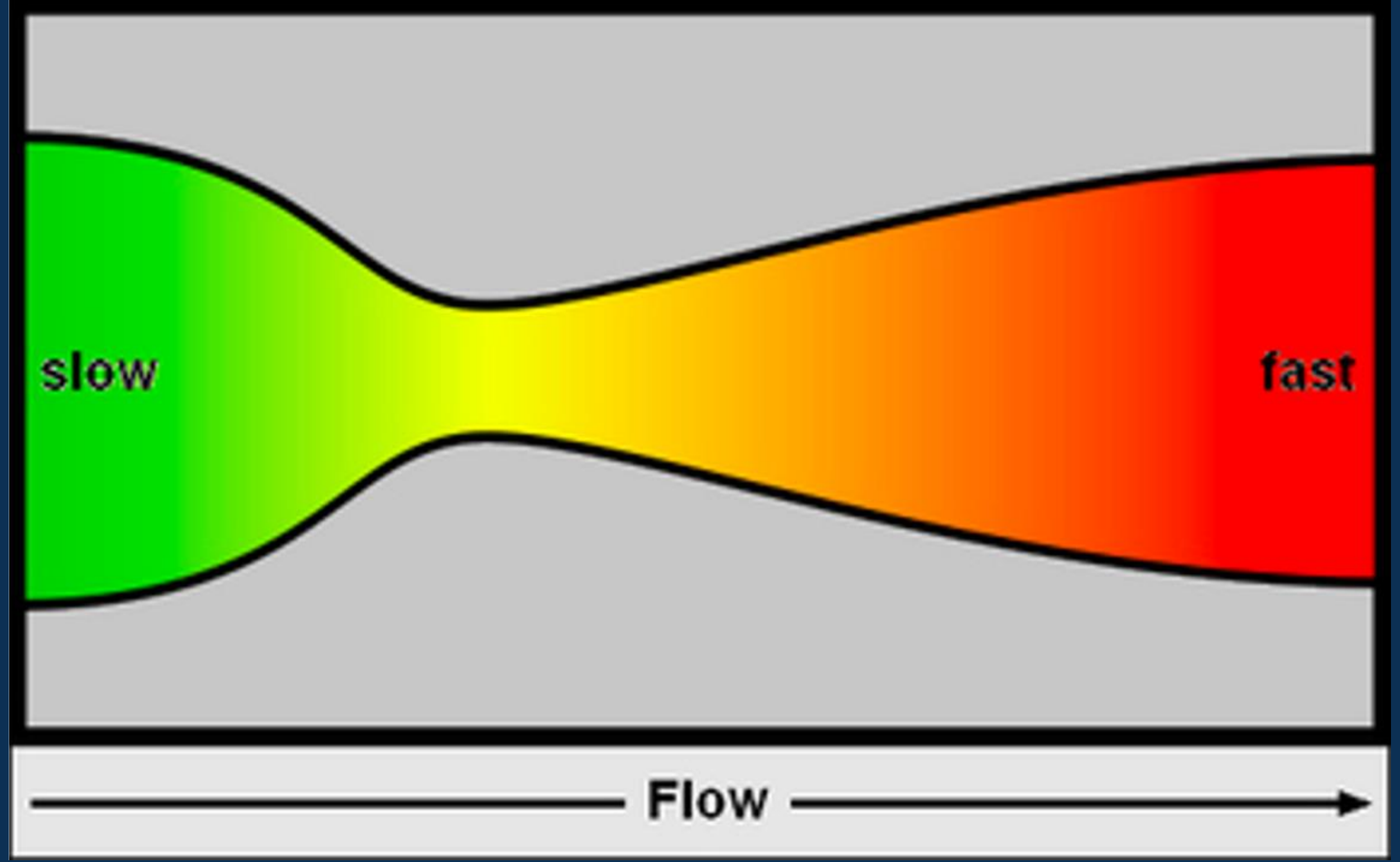
$$U \frac{\partial U}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{g R_s}{r^2}$$

Introducing the Mach number M = U/c

$$\left(M - \frac{1}{M}\right) M' = \frac{2}{r} - \frac{g R_s}{r^2 c^2}$$

$$\frac{1}{2}\left(M^2 - M_0^2\right) - \log\left(\frac{M}{M_0}\right) = 2\log r + \frac{g R_s}{c^2}\left(-1 + \frac{1}{r}\right)$$

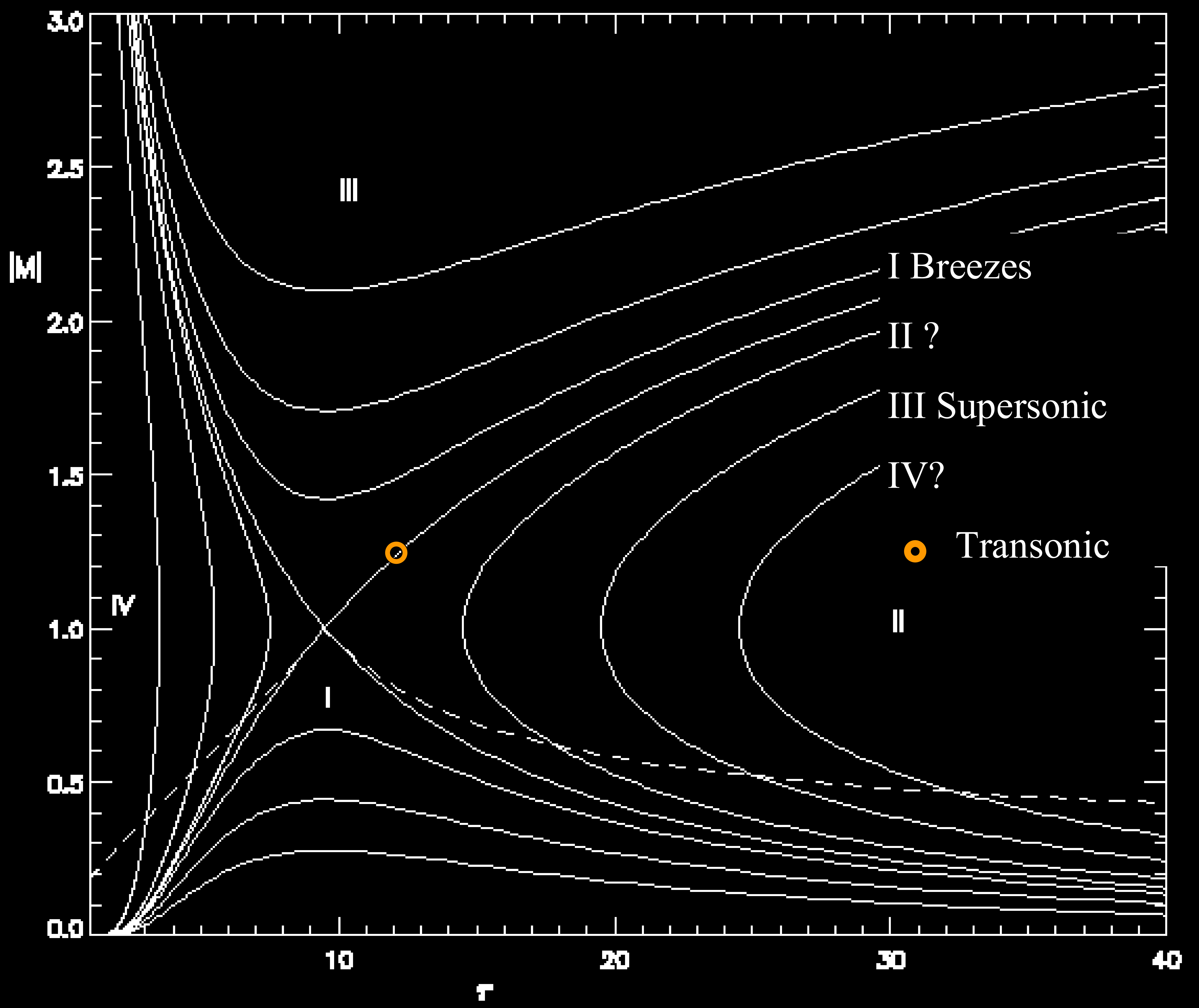
$$\frac{1}{2}\left(M^2 - M_0^2\right) + \log\left(\frac{p}{p_0}\right) = \frac{g R_s}{c^2}\left(-1 + \frac{1}{r}\right)$$



$$\left(U - \frac{V_T^2}{U}\right)\frac{dU}{dr} = \frac{V_T^2}{A}\frac{dA}{dr} - \frac{V_g^2}{2}\frac{1}{r^2}$$

$$\frac{1}{2}\left(M^2 - M_0^2\right) - \log\left(\frac{M}{M_0}\right) = 2\log r + \frac{gR_s}{c^2}\left(-1 + \frac{1}{r}\right)$$

$$\frac{1}{2}\left(M^2 - M_0^2\right) + \log\left(\frac{p}{p_0}\right) = \frac{gR_s}{c^2}\left(-1 + \frac{1}{r}\right)$$



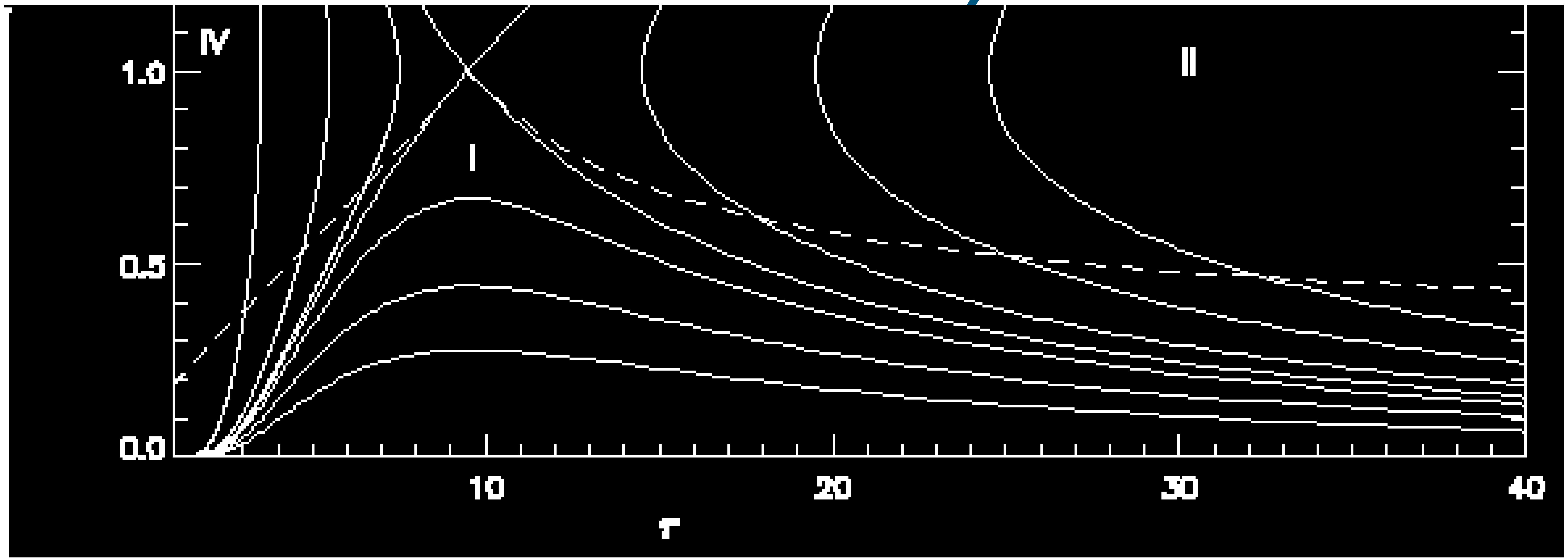
Pressure as a function of distance from the star

$$p = p_0 \exp((M_0^2 - M^2)/2 - gR_s/c^2)$$

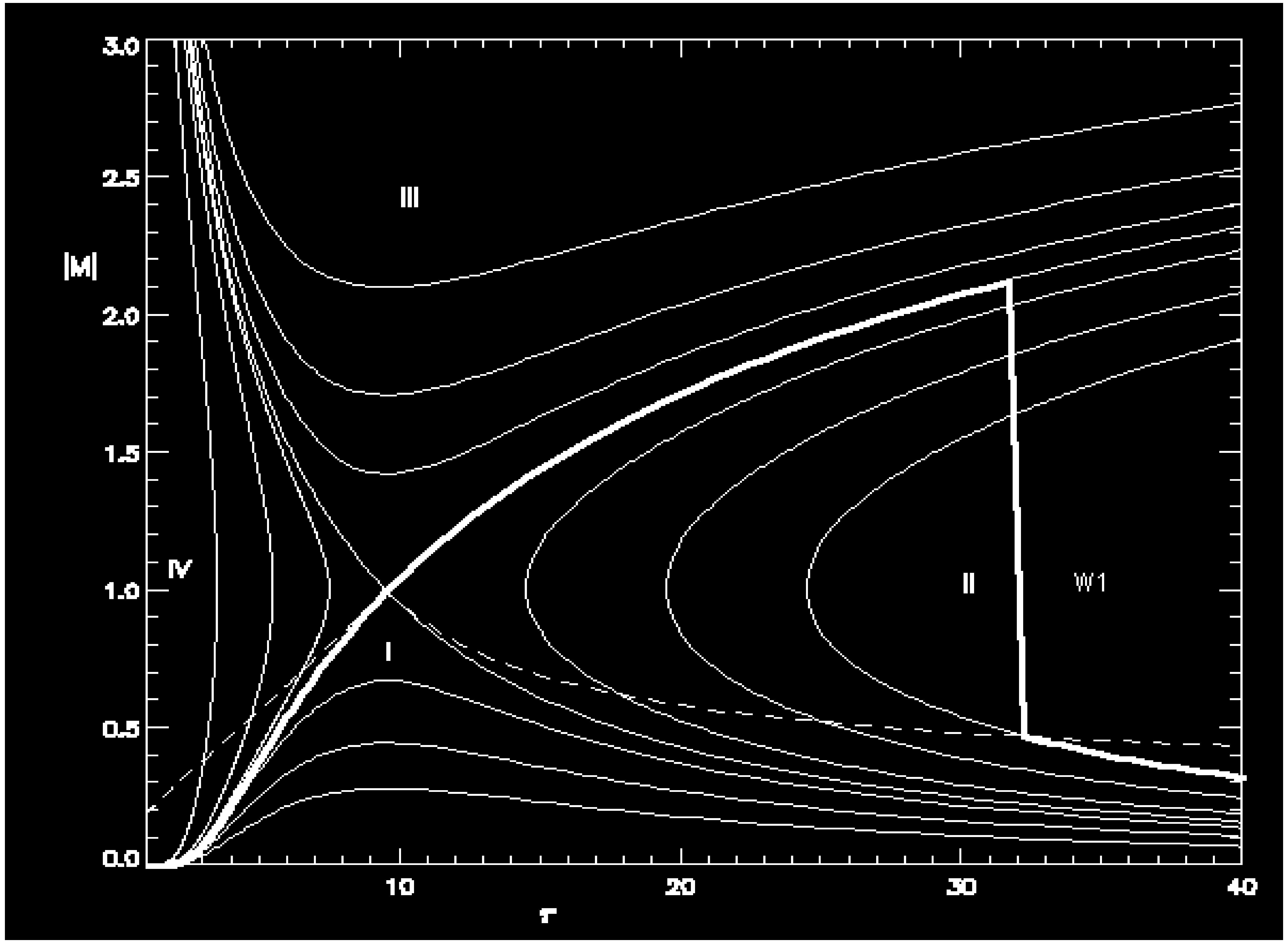
Goes to zero for transonic, for breezes varies from:

$$p_{stat} = p_0 \exp(-gR_s/c^2)$$ **STATIC**

$$p_{crit} = p_0 \exp(M_*^2/2 - gR_s/c^2)$$ **CRITICAL BREEZE (upward + downward transonic)**



Among flows which are subsonic at the atmospheric base the accelerating transonic has the special property that density and pressure tend to zero at large distances: because of the small but finite values of the pressure of the ambient external medium a terminal shock transition connecting to the lower branch of the double valued solutions filling region II will in general be present (McCrea 1956, Holzer and Axford 1970)



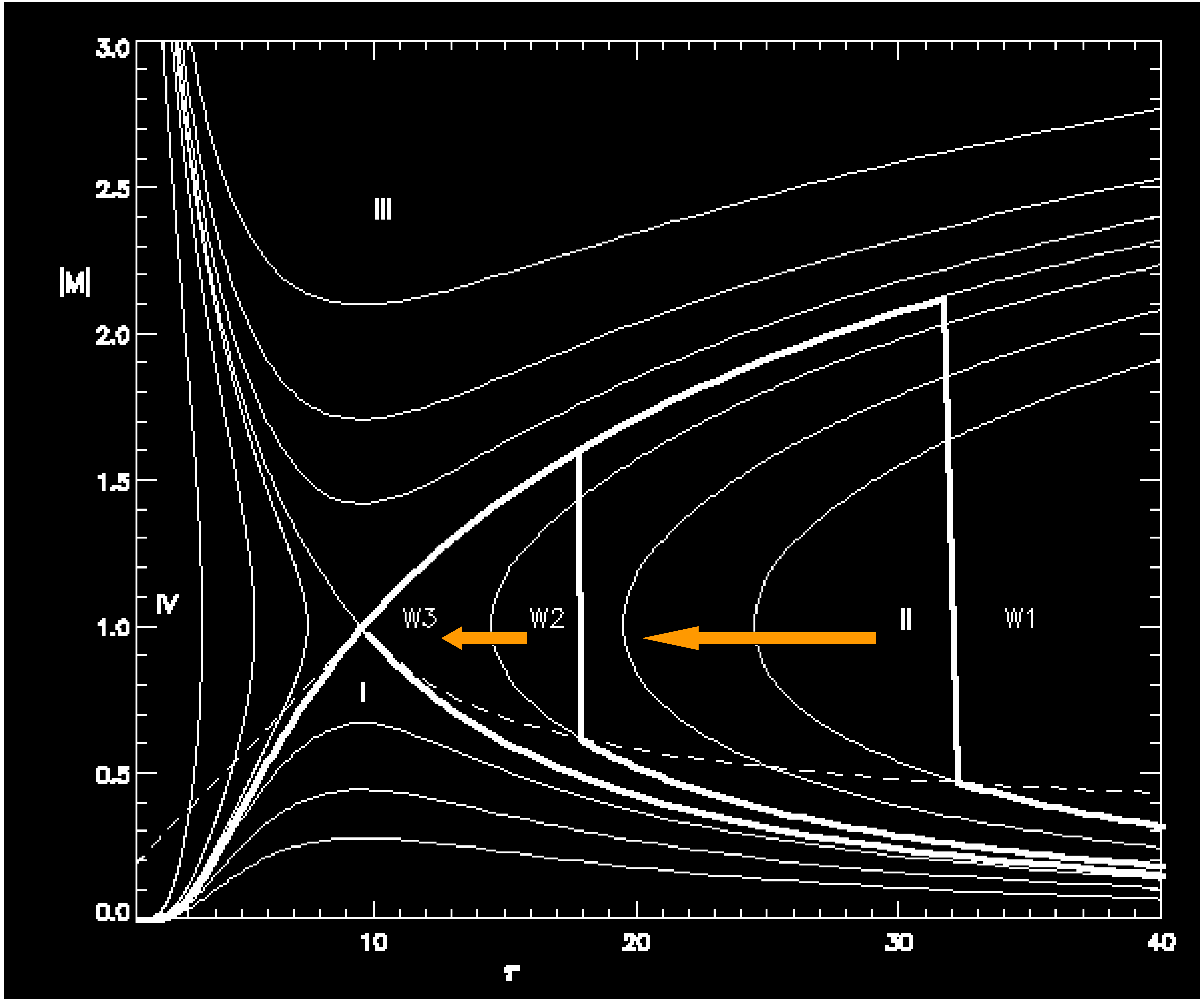
Shock position is determined by the pressure at infinity via the jump conditions.

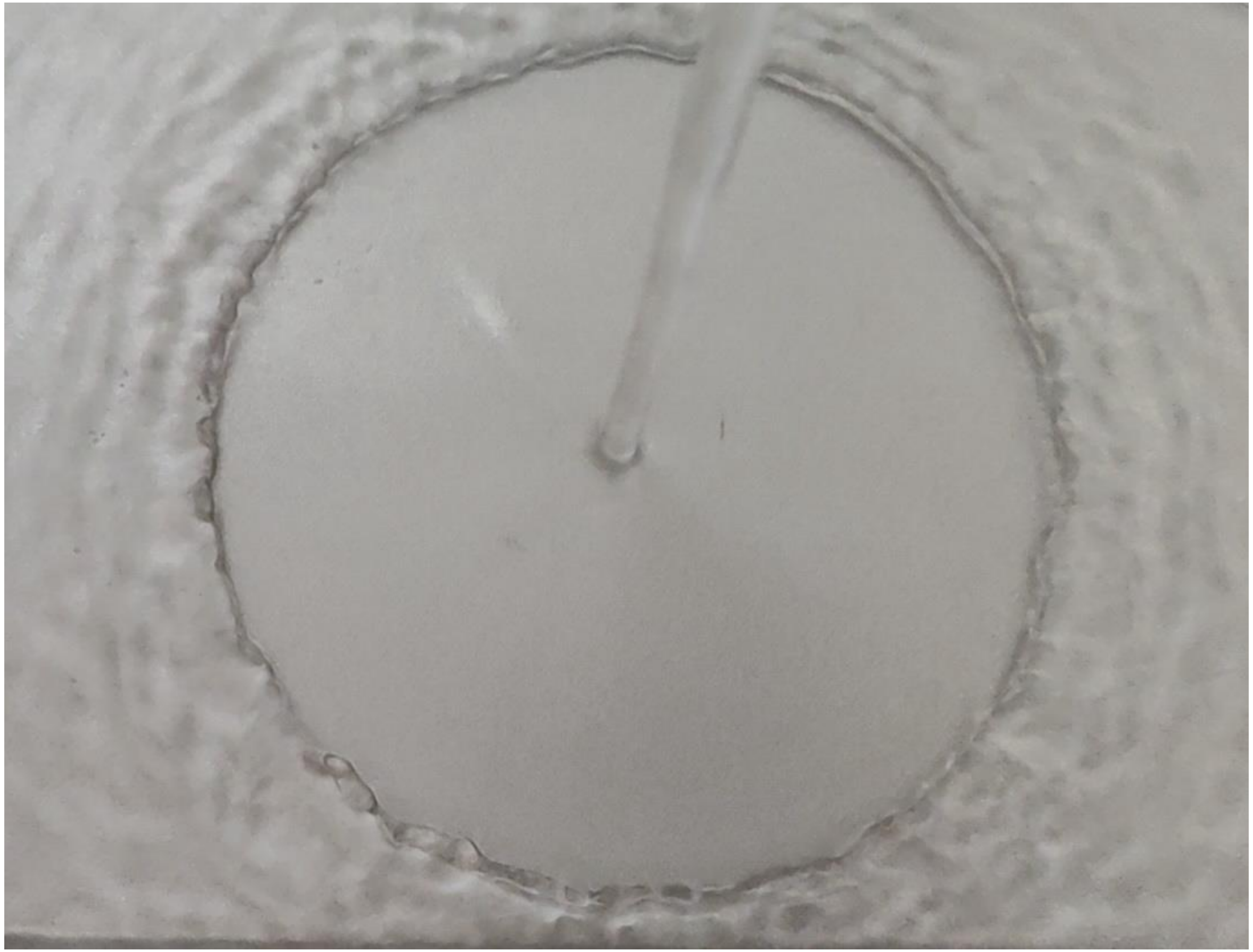
$$\rho^+(M^+)^2 + \rho^+ = \rho^-(M^-)^2 + \rho^-,$$

$$\rho^- M^- = \rho^+ M^+ \longrightarrow M^- M^+ = 1$$

For given base values of the pressure, the position of the shock is uniquely determined by the pressure of the interstellar medium, and the distance from the critical point to the shock decreases as the pressure increases.

As p_∞ increases the shock moves in (obvious + algebra is simple).





SO for intermediate pressures at infinity

there are two solutions: a wind and a breeze.

$$p_{stat} \le p_\infty \le p_{crit}$$

Multiple solutions often mean INSTABILITY . Are breezes REALLY STABLE?

$$y^\pm = \delta U/c \pm \delta p/\rho c^2$$

$$y^\pm = y^\pm(r)\exp(-i(\omega + i\gamma)t)$$

$$(M \pm 1)y^{\pm\prime} - i(\omega + i\gamma)y^\pm + \frac{1}{2}(y^\pm + y^\mp)\frac{M'}{M}(M \mp 1) = 0$$

Look for solutions with vanishing pressure perturbations at R_0 and infinity

For waves in a flowing medium one defines the conserved WAVE ACTION (not energy: waves do work on flow)

$$S = \frac{(M+1)^2}{M}|y^+|^2 - \frac{(M-1)^2}{M}|y^-|^2$$

However instabilities may change the wave action (extract energy from the flow) so equation looks like this:

$$\left[\frac{(M+1)^2}{M}|y^+|^2 - \frac{(M-1)^2}{M}|y^-|^2\right]' + 2\frac{\gamma}{M}\left[(M+1)|y^+|^2 - (M-1)|y^-|^2\right] = 0$$

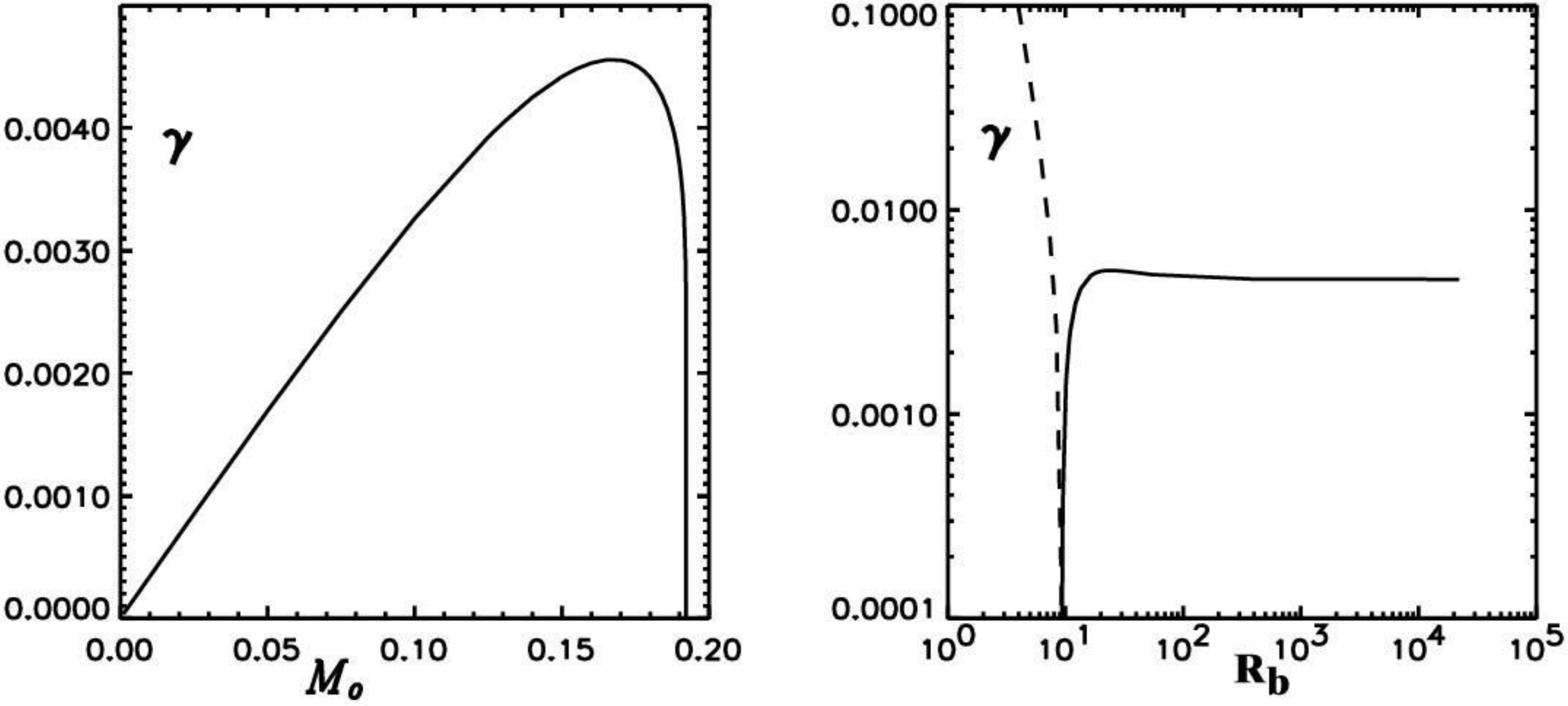
Integrating this equation between 1 and r and imposing the boundary condition that the pressure perturbation vanish (i.e. $y^+ = y^-$) both at the solar surface and at great distances, we find

$$\gamma = \frac{2\left(|y^+|_0^2 - |y^+|_r^2 \right)}{\int_1^r dr\, M^{-1} \left[(M+1)|y^+|^2 - (M-1)|y^-|^2 \right]}$$

But the asymptotics for breezes is obvious…..

$$y^+ \sim \frac{e^{\mp\gamma r}}{r}\left(1 \pm \frac{1}{2\gamma r}\right), \qquad y^- \sim \pm\frac{e^{\mp\gamma r}}{2\gamma r^2}$$

Boundary conditions are satisfied amplitudes tend to zero at great distances, and BREEZES ARE UNSTABLE Growth rate as a function of Base Mach Number



Why are breezes UNSTABLE? Wait a minute, let's look back at the asymptotic pressures ……

$$p_\infty = p_0 \exp\left(M_0^2/2 - g/c^2\right)$$

So pressure at infinity INCREASES with INCREASING BASE MACH NUMBER……

If you have a static atmosphere and increase pressure at infty, you would expect flow to GO IN, NOT OUT – Bondi, 1952

ON SPHERICALLY SYMMETRICAL ACCRETION

H. Bondi

(Received 1951 October 3)

Summary

The special accretion problem is investigated in which the motion is steady and spherically symmetrical, the gas being at rest at infinity. The pressure is taken to be proportional to a power of the density. It is found that the accretion rate is proportional to the square of the mass of the star and to the density of the gas at infinity, and varies inversely with the cube of the velocity of sound in the gas at infinity. The factor of proportionality is not determined by the steady-state equations, though it is confined within certain limits. Arguments are given suggesting that the case physically most likely to occur is that with the maximum rate of accretion.

Bondi Diagram

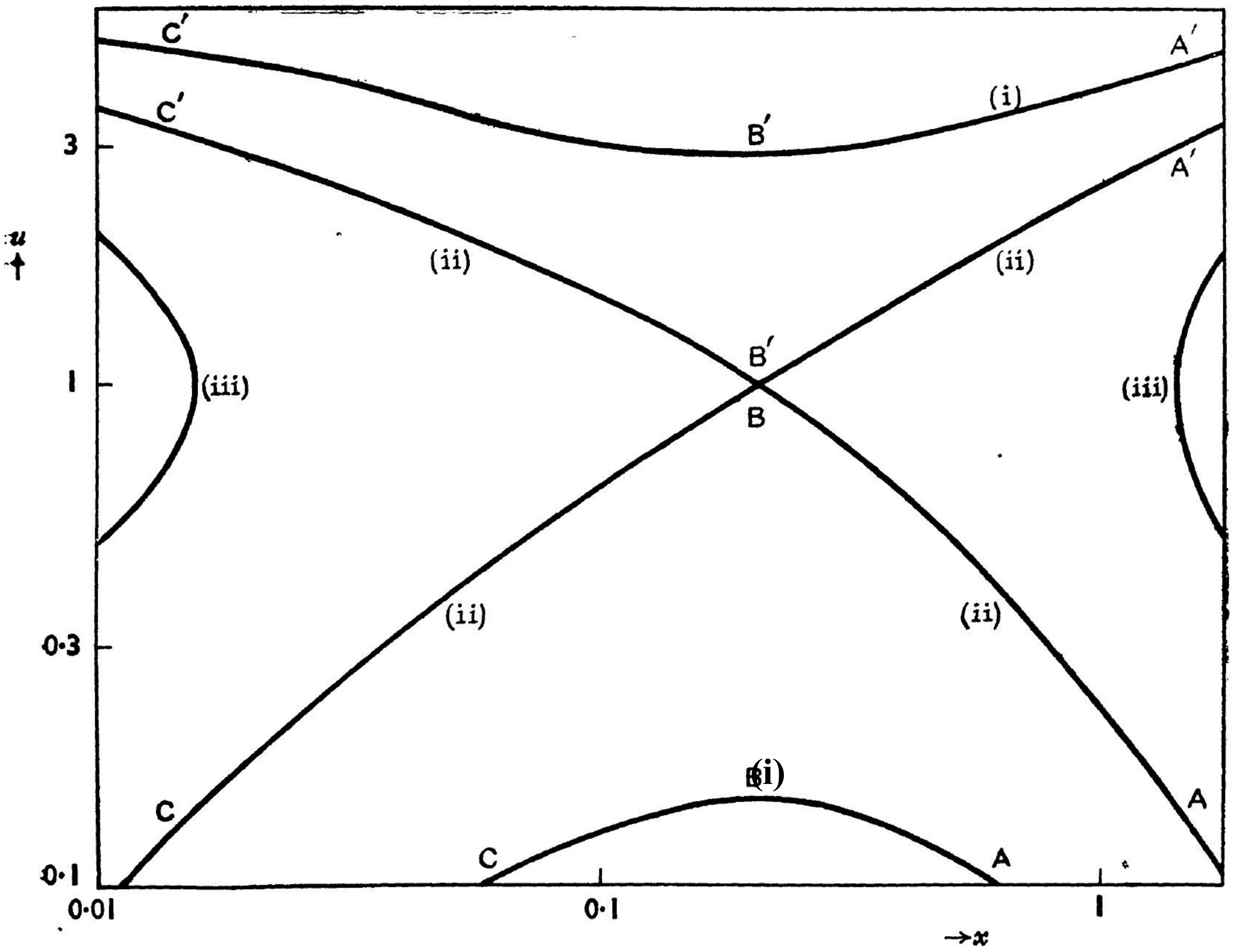


FIG. 2.—*u as function of x for* $\gamma = \frac{7}{5}$.
(i) $\lambda = \frac{1}{4}\lambda_c$;
(ii) $\lambda = \lambda_c$;
(iii) $\lambda = 4\lambda_c$.

SHOCK WAVES IN STEADY RADIAL MOTION UNDER GRAVITY*

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Received May 7, 1956

ABSTRACT

In the steady radial flow of a polytropic gas toward a center of gravitational attraction, it is shown that a standing shock wave can, in general, exist within a certain distance of the center. This possibility elucidates certain features of Bondi's investigation of such radial motion and, in particular, helps to resolve an indeterminacy which he had noted Reasons are given for concluding that the phenomenon may have application to accretion by a binary star.

I. INTRODUCTION

The steady spherically symmetric motion of a gas toward a center of gravitational attraction has been studied by Bondi (1952). The main purpose of the present work is to show that a standing spherical shock wave may occur in such motion. If it does, the motion on the supersonic side is given by a particular solution discovered by Bondi, while the motion on the subsonic side is given by one of a set of solutions which he found but which had appeared to be without physical significance. In this and other ways the work assists in the interpretation of Bondi's investigation.

VIII. NUMERICAL ILLUSTRATION

In order to illustrate the hydrodynamical theory for a value of γ in the range $1 < \gamma < \frac{5}{3}$, it is convenient to use $\gamma = \frac{7}{5}$. This value was also used by Bondi, and Figure 1 reproduces some features of Bondi's Figure 2, though the present calculations have been done afresh.

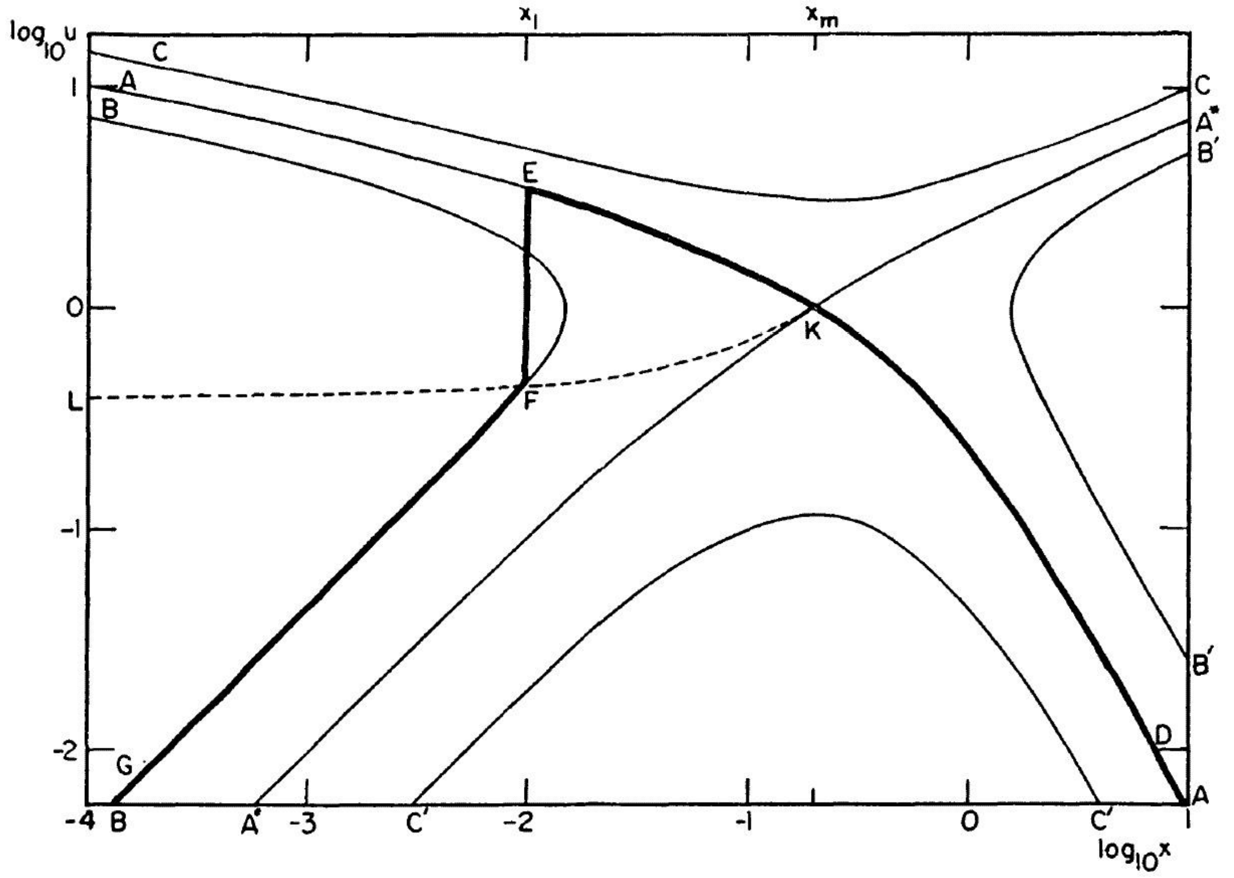
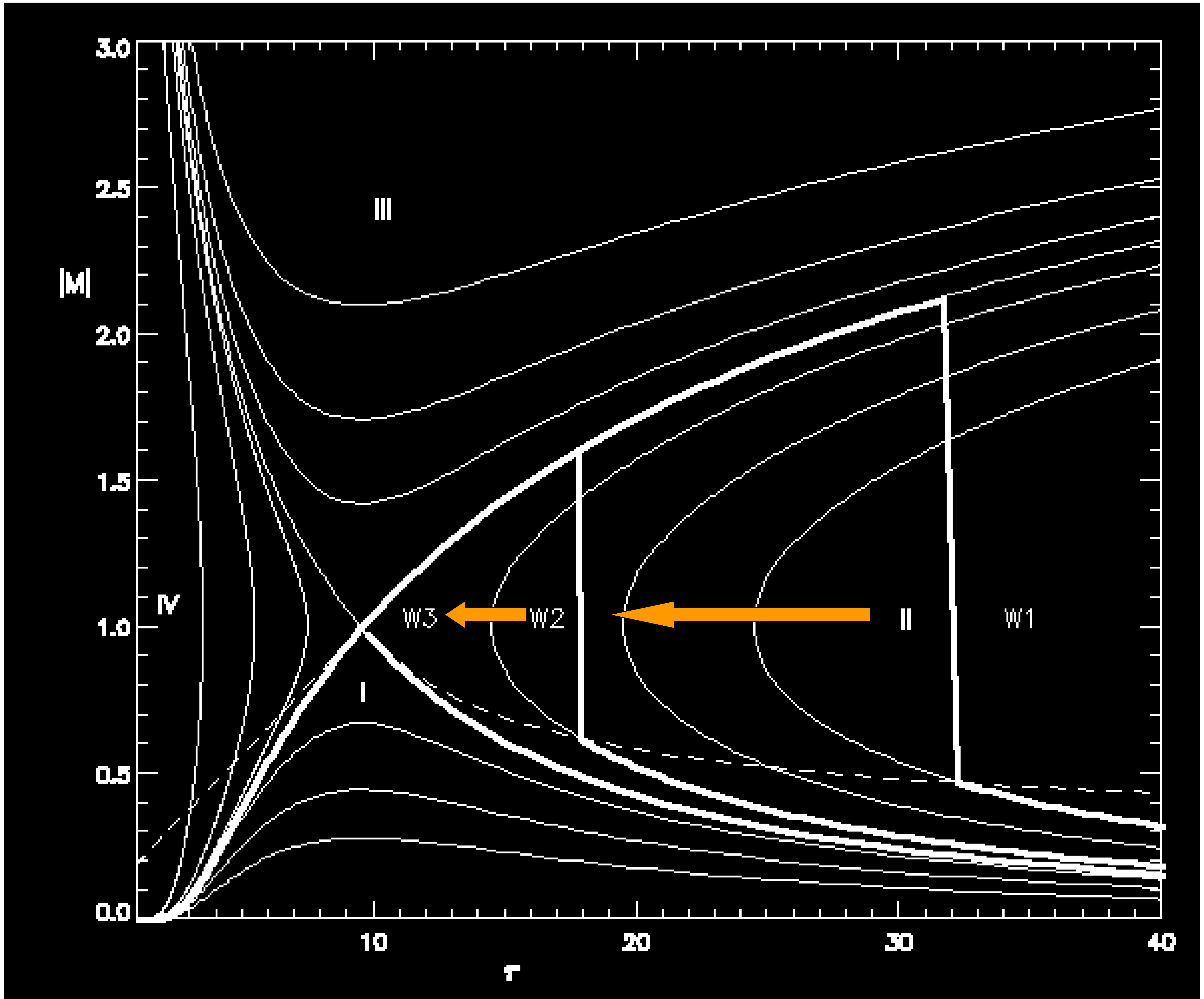


Fig. 1.—Example of motion with shock wave, $\gamma = \frac{7}{5}$. AA, A^*A^*, solution of equation (8 1) for $\kappa = \frac{5}{8}$; BB, $B'B'$, solution for $\kappa \doteqdot 2.77$; $DEFG$, motion with shock EF; LFK, values of u_2 corresponding to u_1 given by AEK; and CC, $C'C'$, solution for $\kappa = \frac{1}{8}$. Abscissa $\log_{10} x$; ordinate $\log_{10} u$.

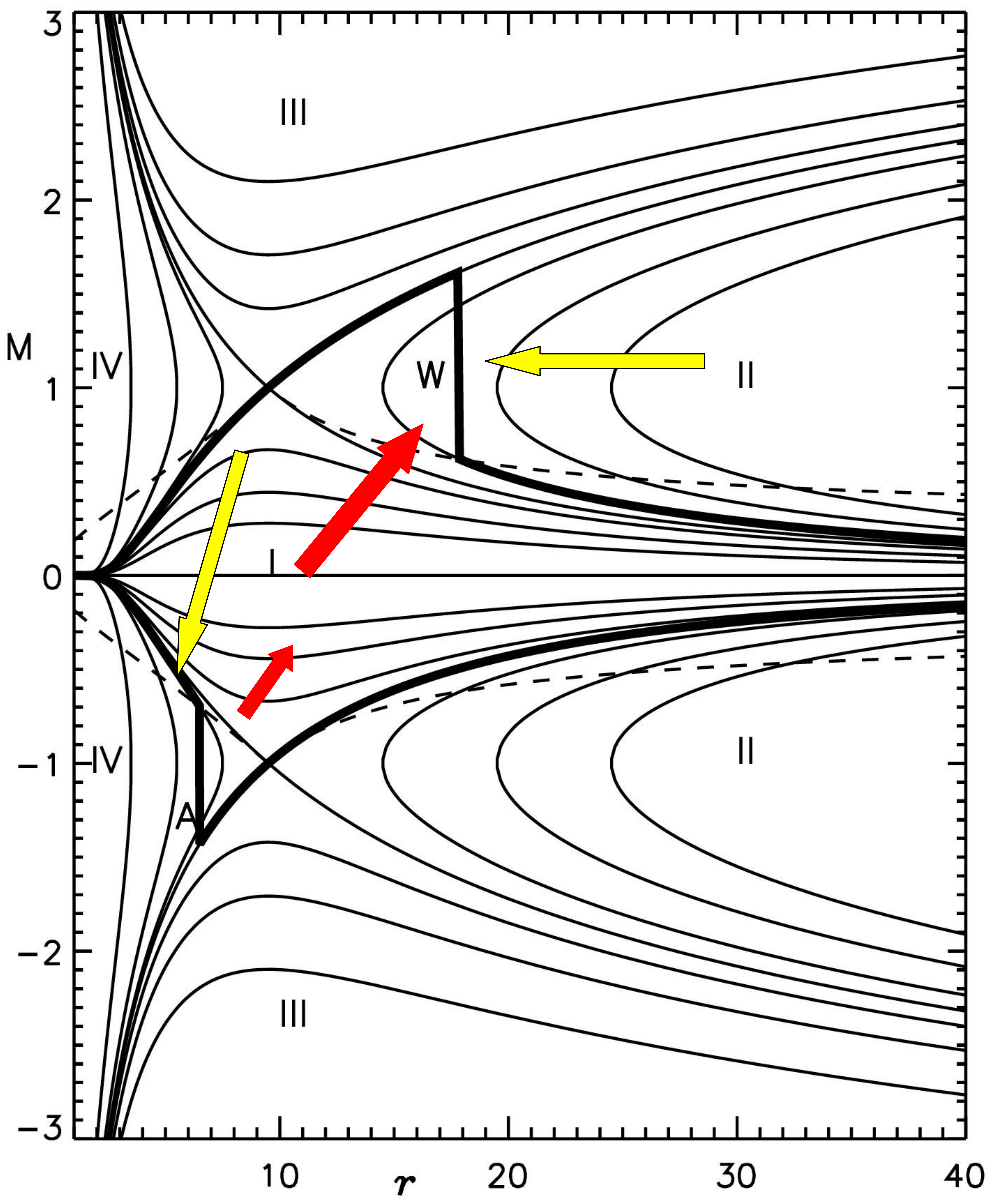
As p_∞ increases above p_{crit} What HAPPENS ??? Can't go to BREEZES ($p_\infty < p_{crit}$ and they're unstable ANYWAY)



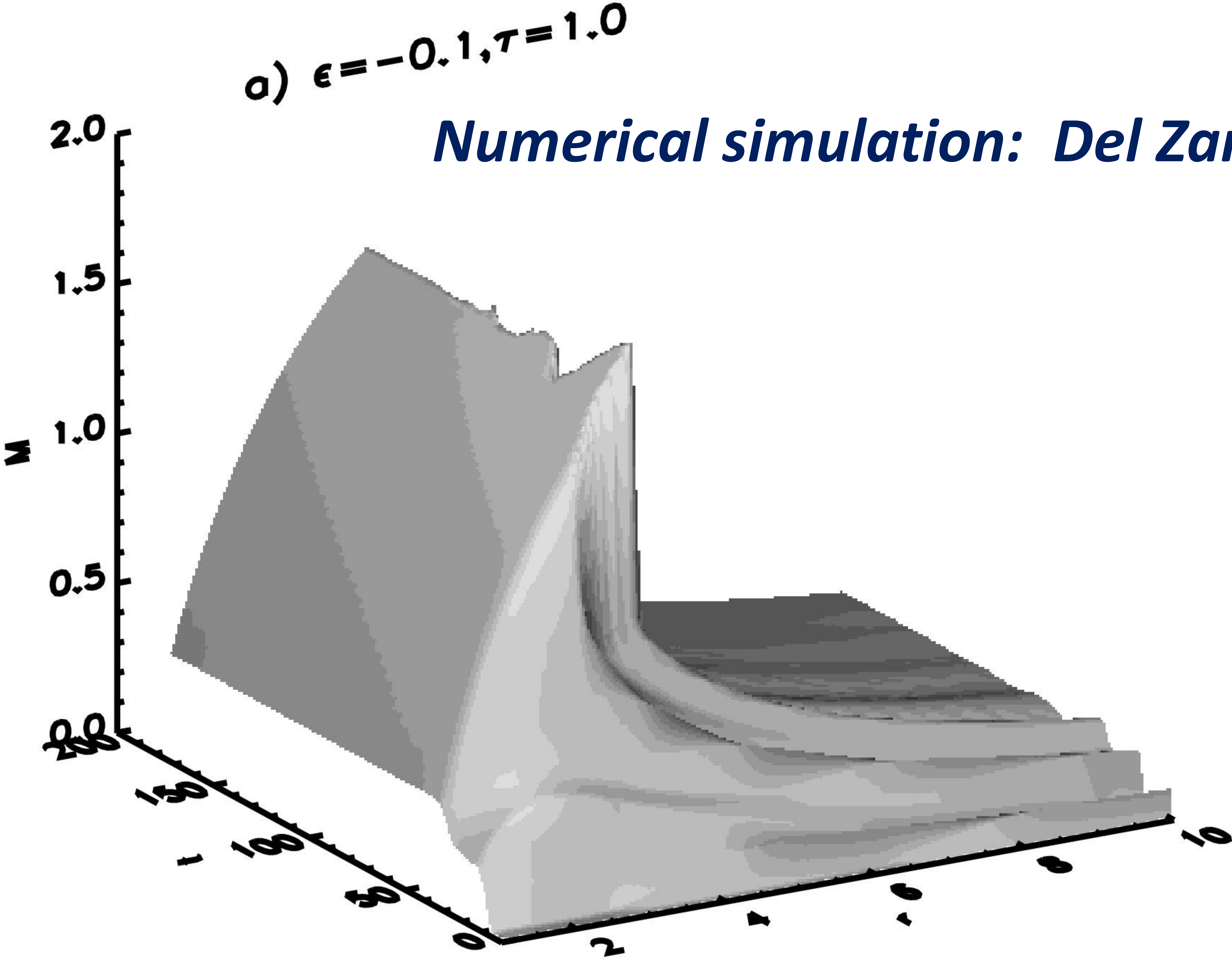
Parker to Bondi And back. Hysteresis cycle (Velli, 1994)

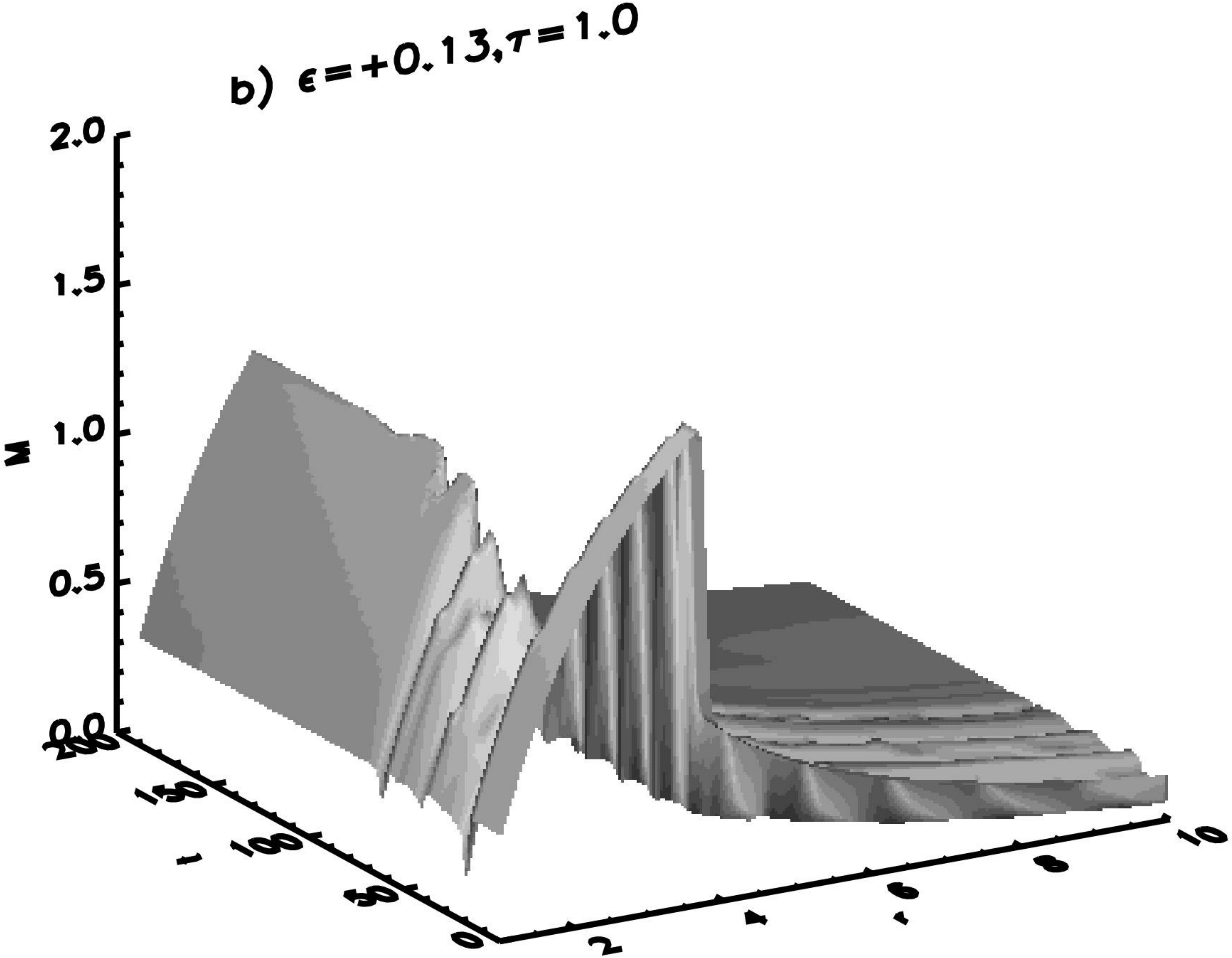
Solves inconsistencies in numerical simulations

Galactic fountains
Supergiant and stellar winds

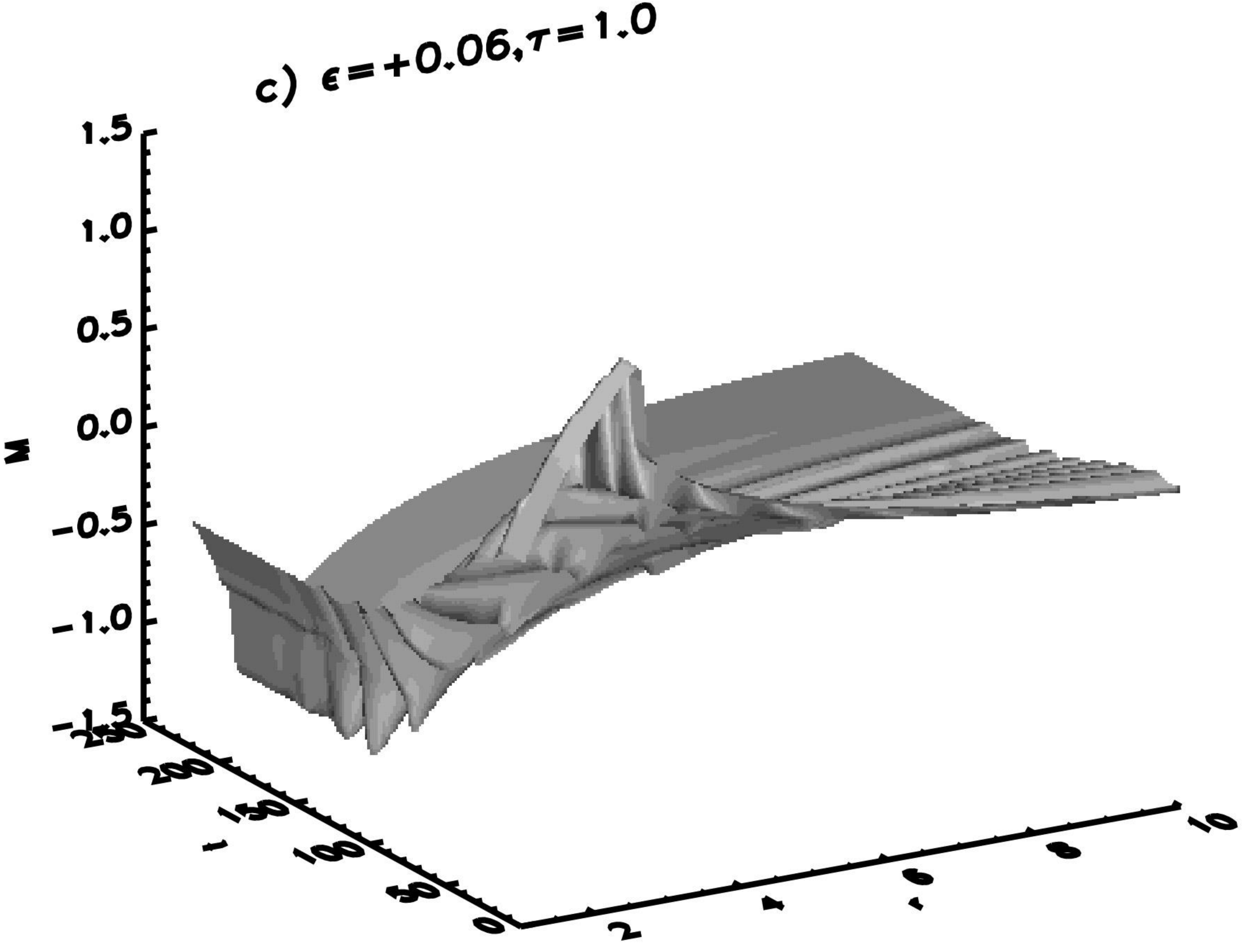


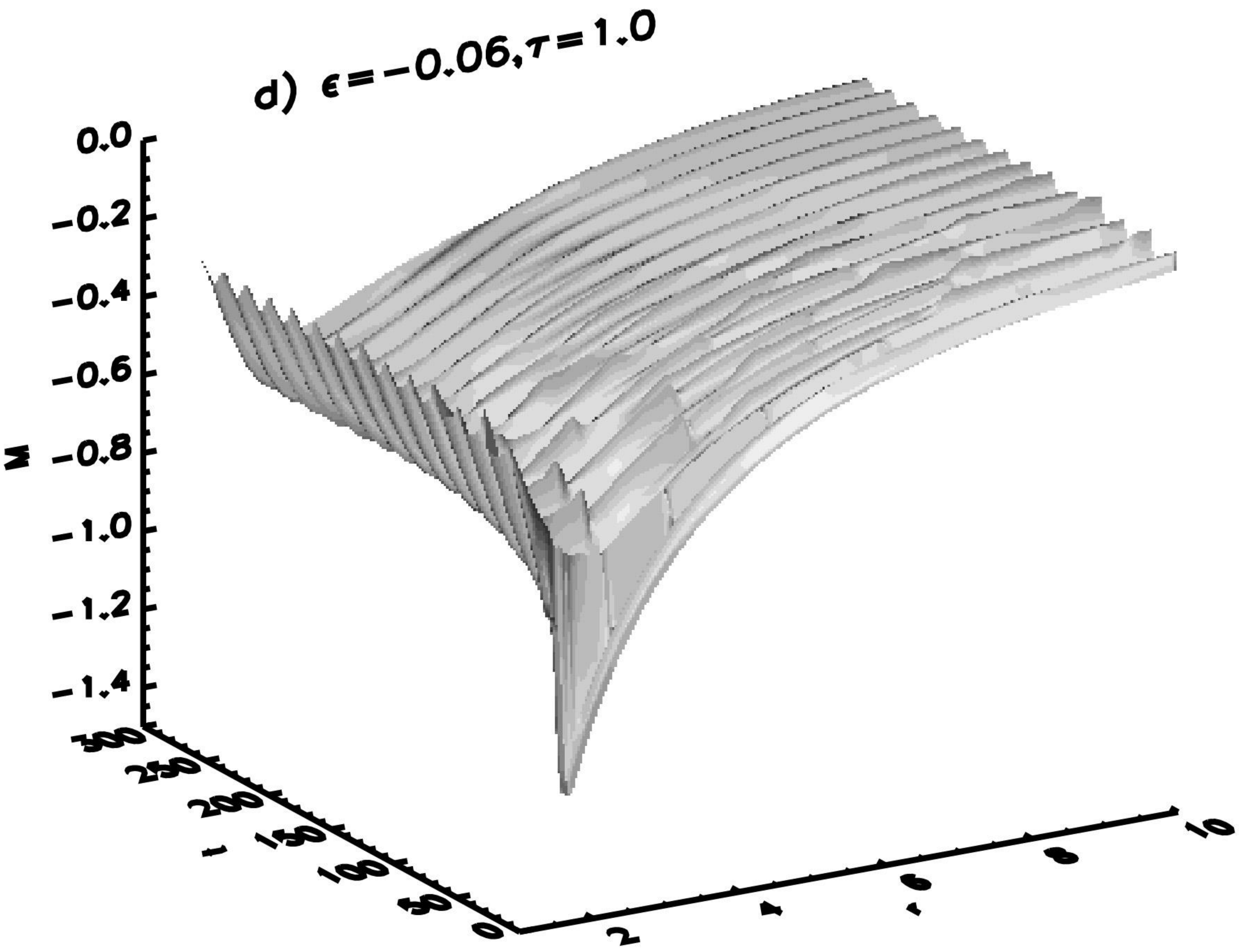
Numerical simulation: Del Zanna et al. 1998





b) ε=+0.13,τ=1.0
2.0
1.5
1.0
0.5
0.0
M
200
150
100
50
0
2
4
6
8
10


c) ϵ=+0.06,τ=1.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
M
250
200
150
t
100
50
0
2
4
6
r
8
10



d) ϵ=−0.06,τ=1.0
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
−1.4
M
300
250
200
150
100
50
0
t
2
4
6
8
10
r

MHD WAVES – SLOW, FAST AND ALFVEN

$$\frac{\partial \rho_0}{\partial t} = \frac{\partial \vec{B}_0}{\partial t} = 0,$$

$$P_0 \rho_0^{-\gamma} = cost.$$

$$0 = -\vec{\nabla} P_0 + \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}_0) \times \vec{B}_0 + \vec{f}_0$$

$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot \rho_0 \vec{U}_1 = 0$$

$$\rho_0 \frac{\partial \vec{U}_1}{\partial t} = -\vec{\nabla}\left(p_1 + \frac{\vec{B}_0 \cdot \vec{B}_1}{4\pi}\right) + \frac{\vec{B}_0 \cdot \vec{\nabla} \vec{B}_1}{4\pi} + \frac{\vec{B}_1 \cdot \vec{\nabla} \vec{B}_0}{4\pi} + \rho_1 \vec{g}$$

$$\frac{\partial p_1}{\partial t} + \vec{U}_1 \cdot \vec{\nabla} p_0 + \gamma p_0 \vec{\nabla} \cdot \vec{U}_1 = 0$$

$$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla} \times (\vec{U}_1 \times \vec{B}_0)$$

$$\vec{r}_1 = \vec{r}_0 + \xi(\vec{r}_0, t) \qquad \xi(\vec{r}_0, t)$$

$$\vec{U}_1 = \frac{d\vec{r}_1}{dt} = \frac{d\vec{\xi}}{dt} \simeq \frac{\partial \vec{\xi}}{\partial t}$$

$$\frac{\partial}{\partial t}(\rho_1 + \vec{\nabla} \cdot (\rho_0 \vec{\xi})) = 0, \qquad \rho_1 + \vec{\nabla} \cdot (\rho_0 \vec{\xi}) = C = 0$$

$$\rho_1 = -\vec{\nabla} \cdot (\rho_0 \vec{\xi})$$

$$\vec{B}_1 = \vec{\nabla} \times \left(\vec{\xi} \times \vec{B}_0\right)$$

$$p_1 = -\vec{\xi} \cdot \vec{\nabla} p_0 - \gamma p_0 \vec{\nabla} \cdot \vec{\xi}$$

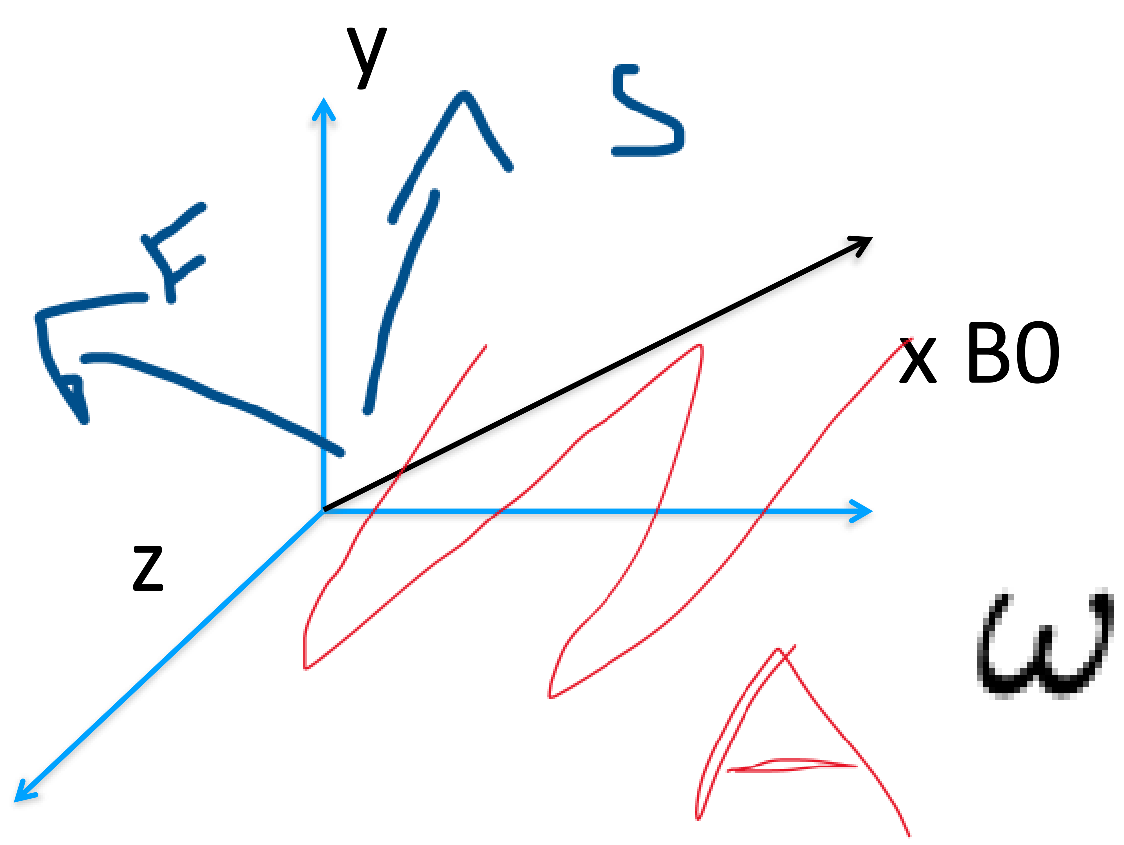
$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \vec{\nabla}(\vec{\xi} \cdot \vec{\nabla} p_0) + \vec{\nabla}(\gamma p_0 \vec{\nabla} \cdot \vec{\xi}) - \vec{g}\, \vec{\nabla} \cdot (\rho_0 \vec{\xi}) +$$
$$\frac{1}{4\pi} \vec{\nabla} \times \left(\vec{\nabla} \times (\vec{\xi} \times \vec{B}_0) \right) \times \vec{B}_0 + \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}_0) \times (\vec{\xi} \times \vec{B}_0)$$
$$= \vec{F}(\vec{\xi}).$$

$$\vec{F}(\vec{\xi})$$

$$\vec{\xi} = \vec{Y}\exp(-i\omega t + i\vec{k}\cdot\mathbf{r})$$

$$-\omega^2\vec{Y} = -c_s^2\vec{k}(\vec{k}\cdot\vec{Y}) + c_a^2 k_\parallel Y_\parallel \vec{k} - c_a^2\vec{k}_\perp(\vec{k}\cdot\vec{Y}) - c_a^2 k_\parallel^2 \vec{Y}$$

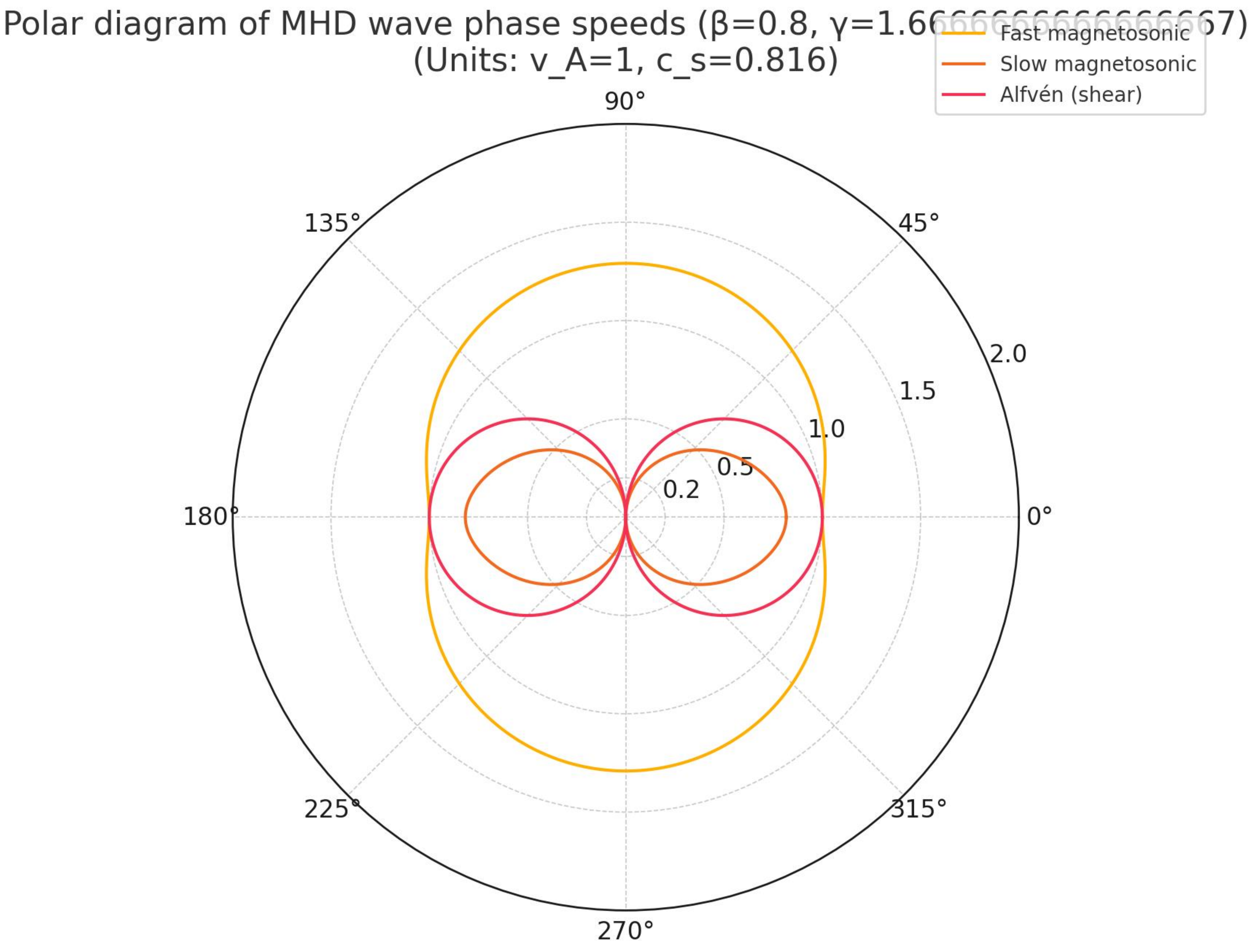
$$c_s = \left(\frac{\gamma p_0}{\rho_0}\right)^{\frac{1}{2}}, \quad c_a = \left(\frac{B_0^2}{4\pi\rho_0}\right)^{\frac{1}{2}}$$

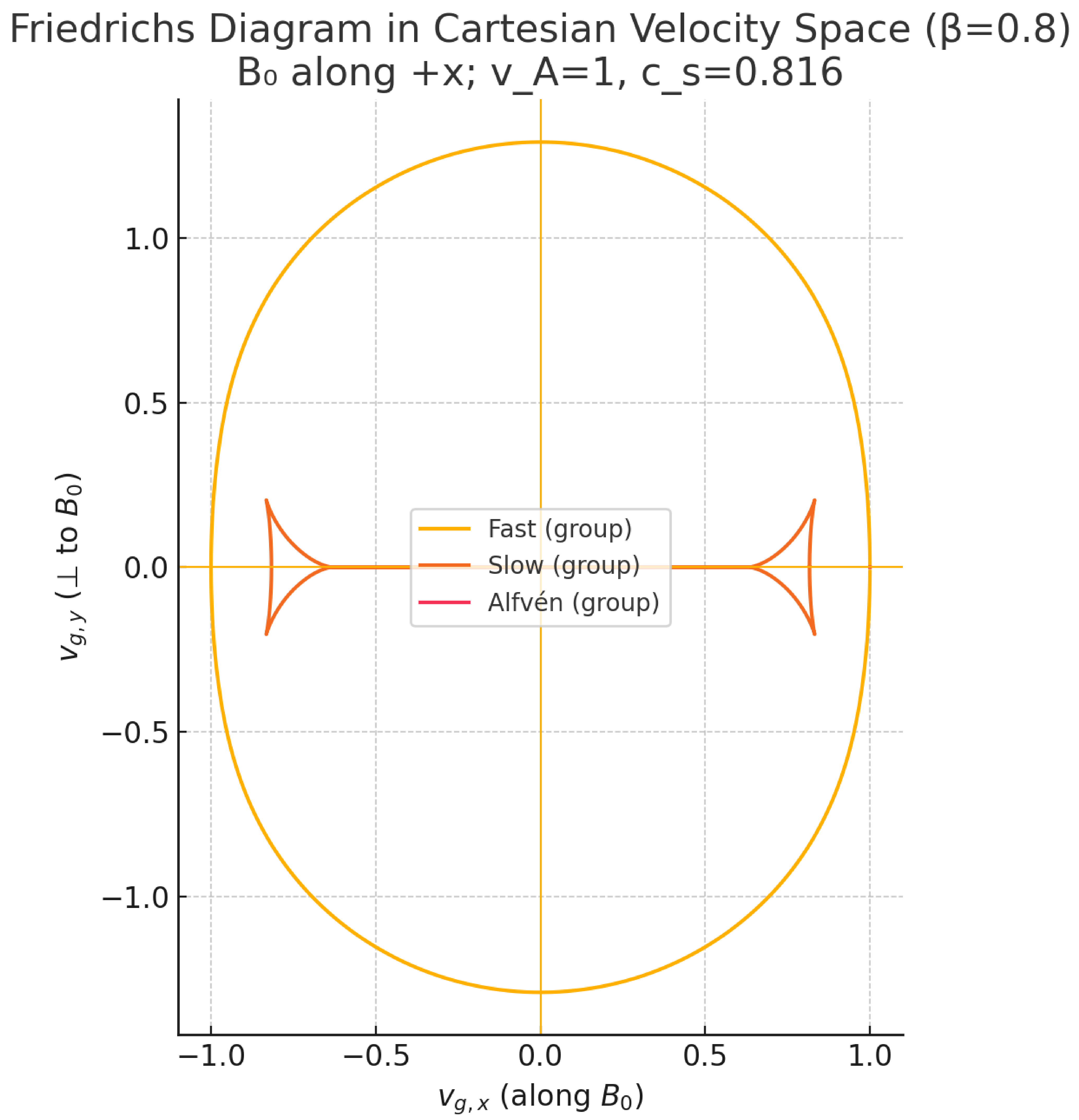


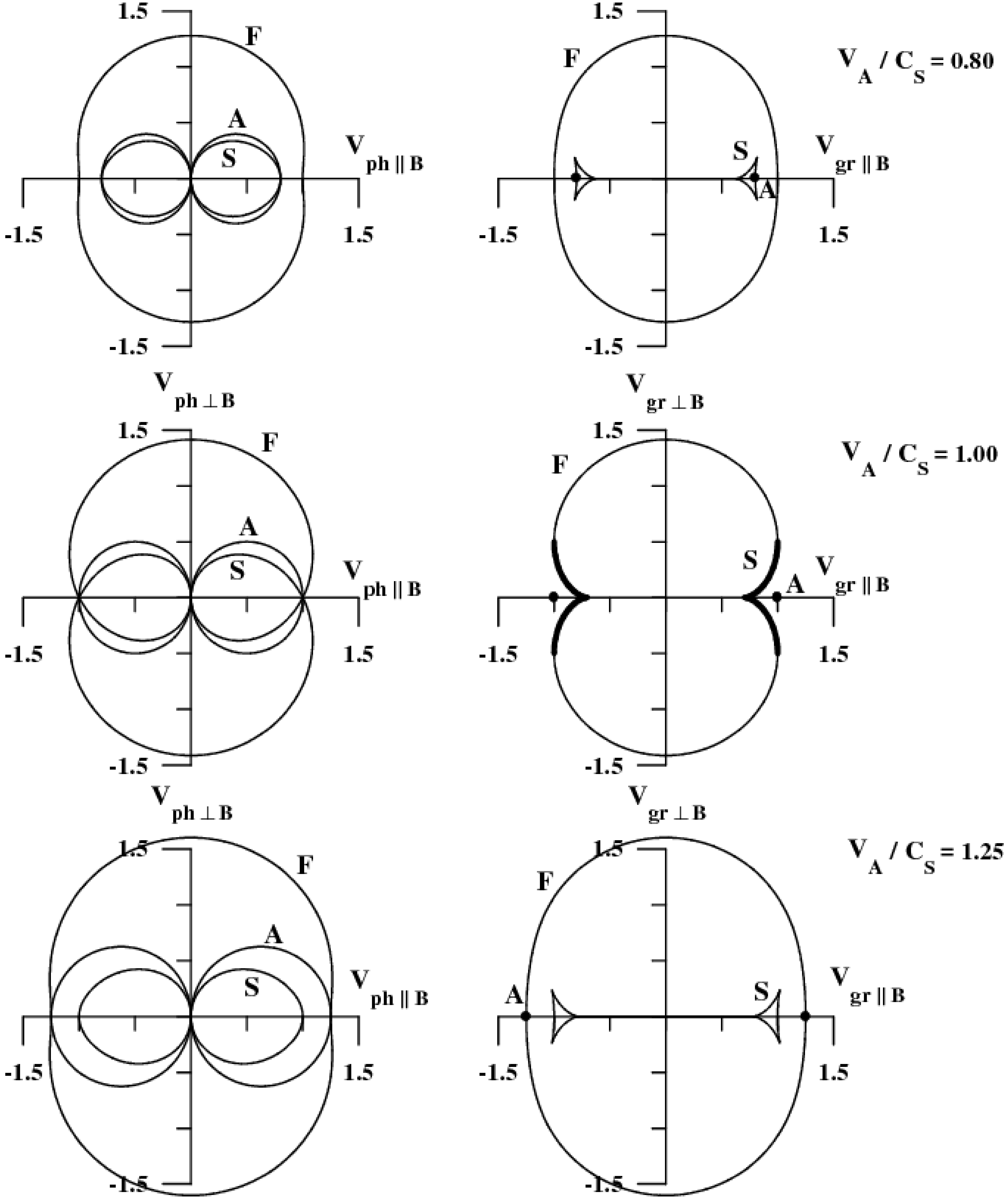
$$\omega^2 = (\vec{k}\cdot\vec{V}_A)^2$$

$$\frac{\omega^2}{k^2} = \frac{c_s^2 + v_A^2 \pm \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 cos^2\theta}}{2}$$

$$u_z = \mp\frac{b_z}{\sqrt{4\pi\rho}}$$


Polar diagram of MHD wave phase speeds (β=0.8, γ=1.6666666666666667)
(Units: v_A=1, c_s=0.816)
Fast magnetosonic
Slow magnetosonic
Alfvén (shear)
90°
135°
45°
180°
0°
225°
315°
270°
0.2
0.5
1.0
1.5
2.0


Friedrichs Diagram in Cartesian Velocity Space (β=0.8)
B₀ along +x; v_A=1, c_s=0.816
Fast (group)
Slow (group)
Alfvén (group)
$v_{g,y}$ (⊥ to B_0)
$v_{g,x}$ (along B_0)
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0



$V_A / C_S = 0.80$
$V_A / C_S = 1.00$
$V_A / C_S = 1.25$
$V_{ph \parallel B}$
$V_{ph \perp B}$
$V_{gr \parallel B}$
$V_{gr \perp B}$
F
A
S
1.5
-1.5

Assuming $p, B^2/2\mu_0 = const$ and the velocity is full incompresible $\nabla \cdot \vec{u} = 0$, the MHD equations reduces to:

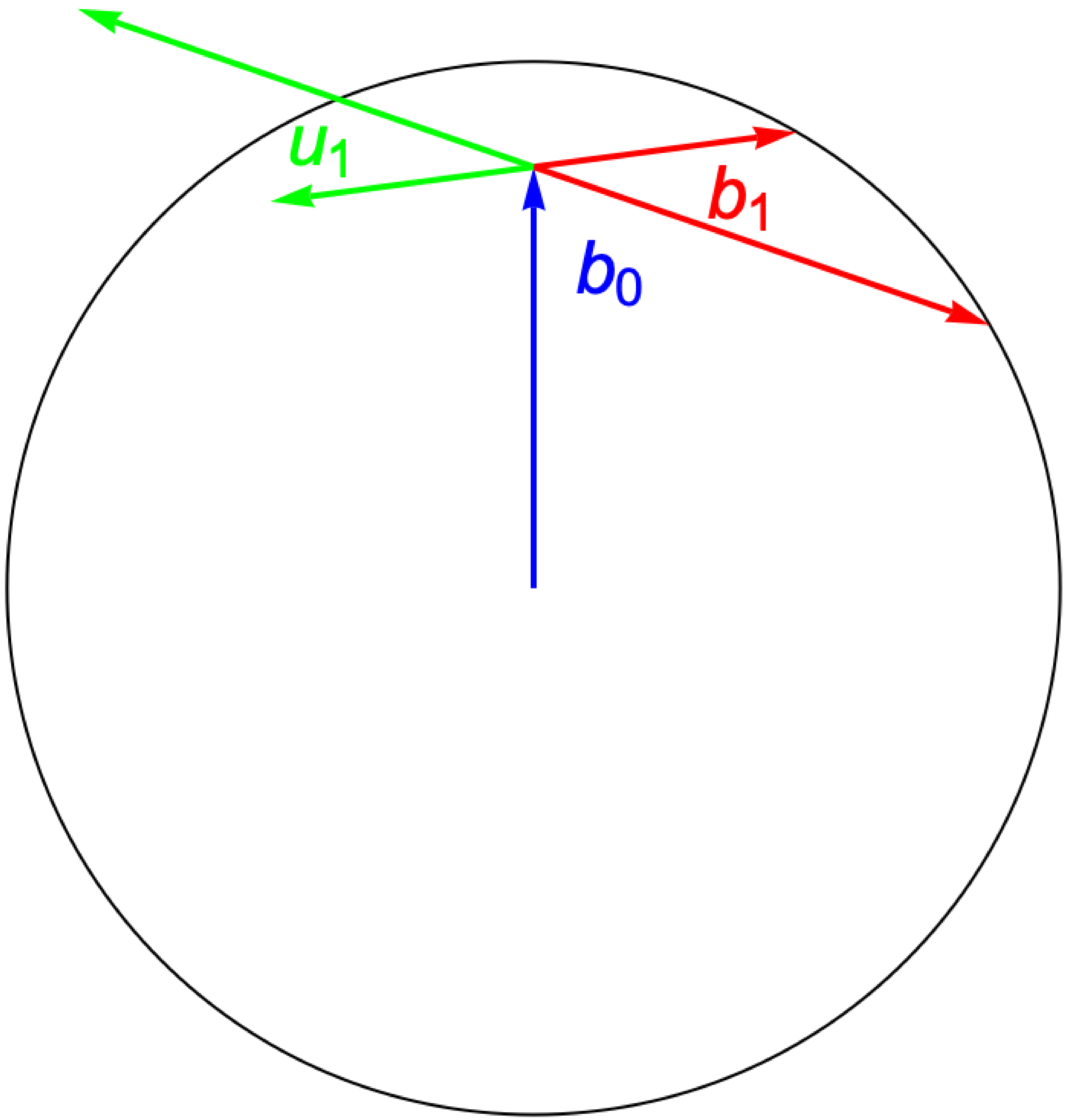
$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} &= \nabla \times (\vec{u} \times \vec{B}) \\ \rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] &= -\nabla p + \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B} \\ &= \underbrace{-\nabla\left(p + \frac{B^2}{2\mu_0}\right)}_{=0} + \frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B} \end{aligned} \tag{1}$$

This can be further reduced to:

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} &= \nabla \times (\vec{u} \times \vec{B}) \\ &= \underbrace{\vec{u}(\nabla\cdot\vec{B})}_{=0} - \underbrace{\vec{B}(\nabla\cdot\vec{u})}_{=0} + (\vec{B}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{B} \\ \frac{\partial \vec{u}}{\partial t} &= -(\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\mu_0 \rho}(\vec{B}\cdot\nabla)\vec{B} \end{aligned} \tag{2}$$

Now we can rewrite $\vec{u} = \vec{u}_0 + \vec{u}_1$ and $\vec{B} = \vec{B}_0 + \vec{B}_1$, and in Alfvén units, the magnetic fields can be rewritten as: $\vec{b} = \vec{b}_0 + \vec{b}_1$, where $\vec{b} = \vec{B}/\sqrt{\mu_0 \rho}$. The equations above now reduces to:

$$\begin{aligned} \frac{\partial \vec{b}}{\partial t} &= -(\vec{u}\cdot\nabla)\vec{u} + (\vec{b}\cdot\nabla)\vec{b} \\ \frac{\partial \vec{u}}{\partial t} &= (\vec{b}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{b} \end{aligned} \tag{3}$$



and then:

$$\begin{aligned}\frac{\partial \vec{b}_1}{\partial t} &= -(\vec{u}_1 \cdot \nabla)\vec{u}_1 - (\vec{u}_0 \cdot \nabla)\vec{u}_1 + (\vec{b}_0 \cdot \nabla)\vec{b}_1 + (\vec{b}_1 \cdot \nabla)\vec{b}_1 \\ \frac{\partial \vec{u}_1}{\partial t} &= (\vec{b}_0 \cdot \nabla)\vec{u}_1 + (\vec{b}_1 \cdot \nabla)\vec{u}_1 - (\vec{u}_1 \cdot \nabla)\vec{b}_1 - (\vec{u}_0 \cdot \nabla)\vec{b}_1\end{aligned} \tag{5}$$

Now obviously there are two solutions to this system:

$$\vec{u}_1 = \pm\vec{b}_1 \tag{6}$$

We may now apply the outward propagating solution: $\vec{u}_1 = -\vec{b}_1$, the equations simply reduces to:

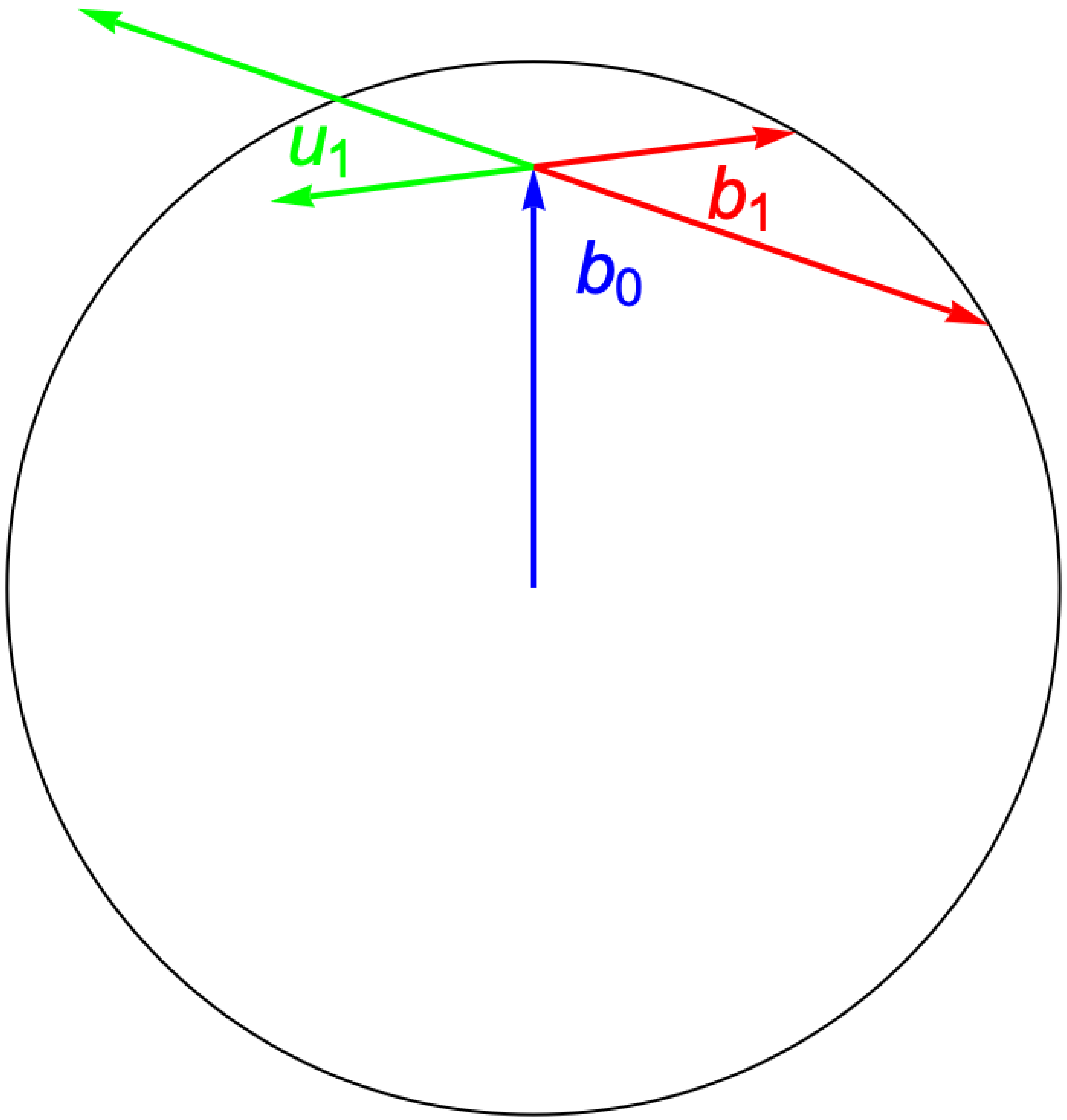
$$\frac{\partial \vec{b}_1}{\partial t} + \left[(\vec{b}_0 + \vec{u}_0) \cdot \nabla\right]\vec{b}_1 = 0 \tag{7}$$

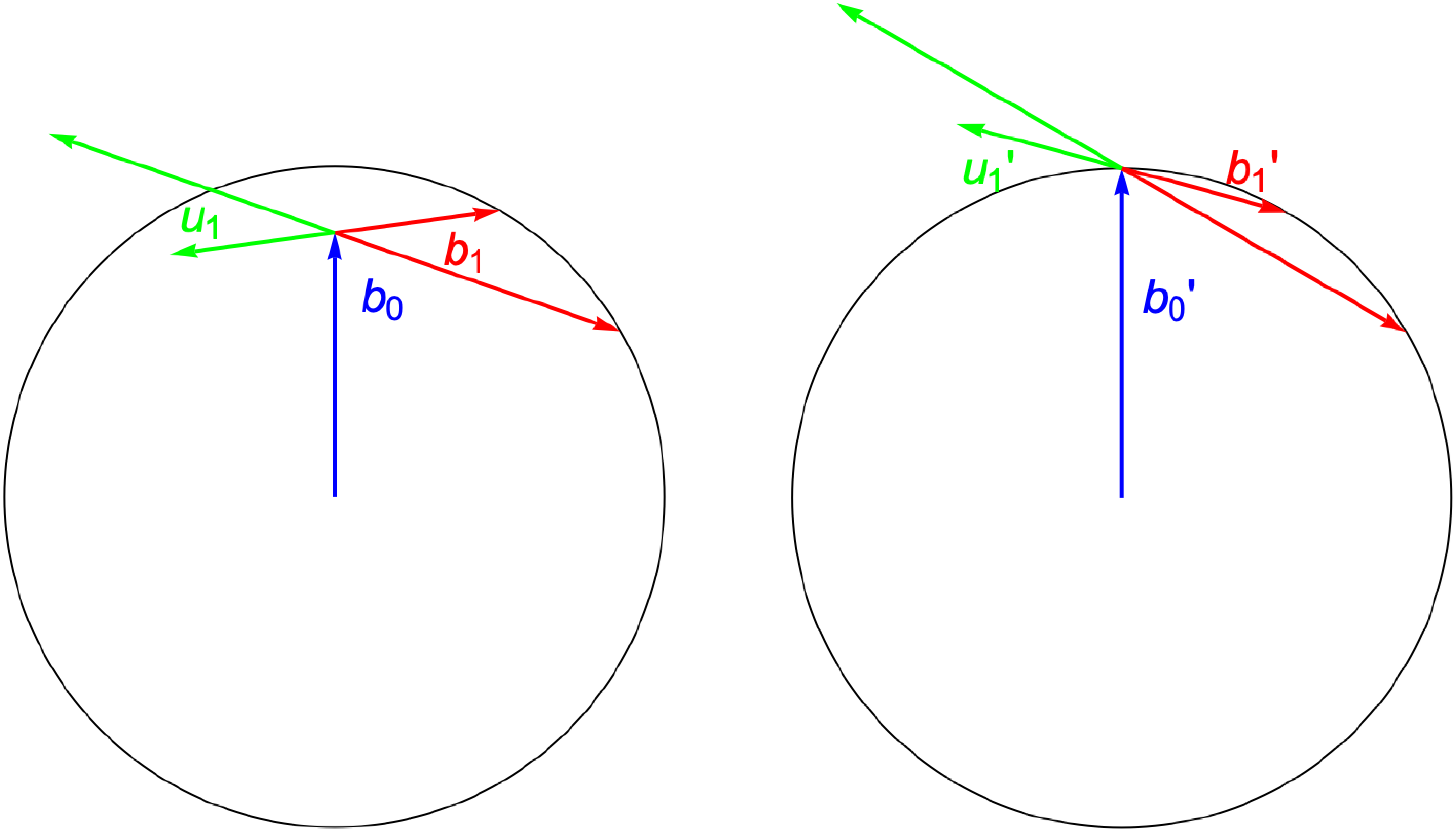
This is the *Spherically Polarized Alfvén Wave*. However, the choice of $\vec{b}_0$ is arbitrary. The most natural choice is to choose $\vec{b}_0$ and $\vec{u}_0$ such that $\langle\vec{b}_1\rangle = 0$ and $\langle\vec{u}_1\rangle = 0$.

However, we can make another choice such that $\vec{b}'_0 = |\vec{B}|\hat{b}_0 = \vec{b}_0 + \Delta\vec{b}_0$. In this case, the perterbation $\vec{b}'_1 = \vec{b}_1 - \Delta\vec{b}_0$. The velocity perturbation now becomes: $\vec{u}'_1 = -\vec{b}'_1 = -\vec{b}_1 + \Delta\vec{b}_0$, and therefore the new background speed $\vec{u}'_0 = \vec{u}_0 - \Delta\vec{b}_0$, such that $\langle\vec{u}\rangle = \vec{u}_0$. Consequently, the new group velocity:

$$\vec{b}'_0 + \vec{u}'_0 = \vec{b}_0 + \vec{u}_0 \tag{8}$$

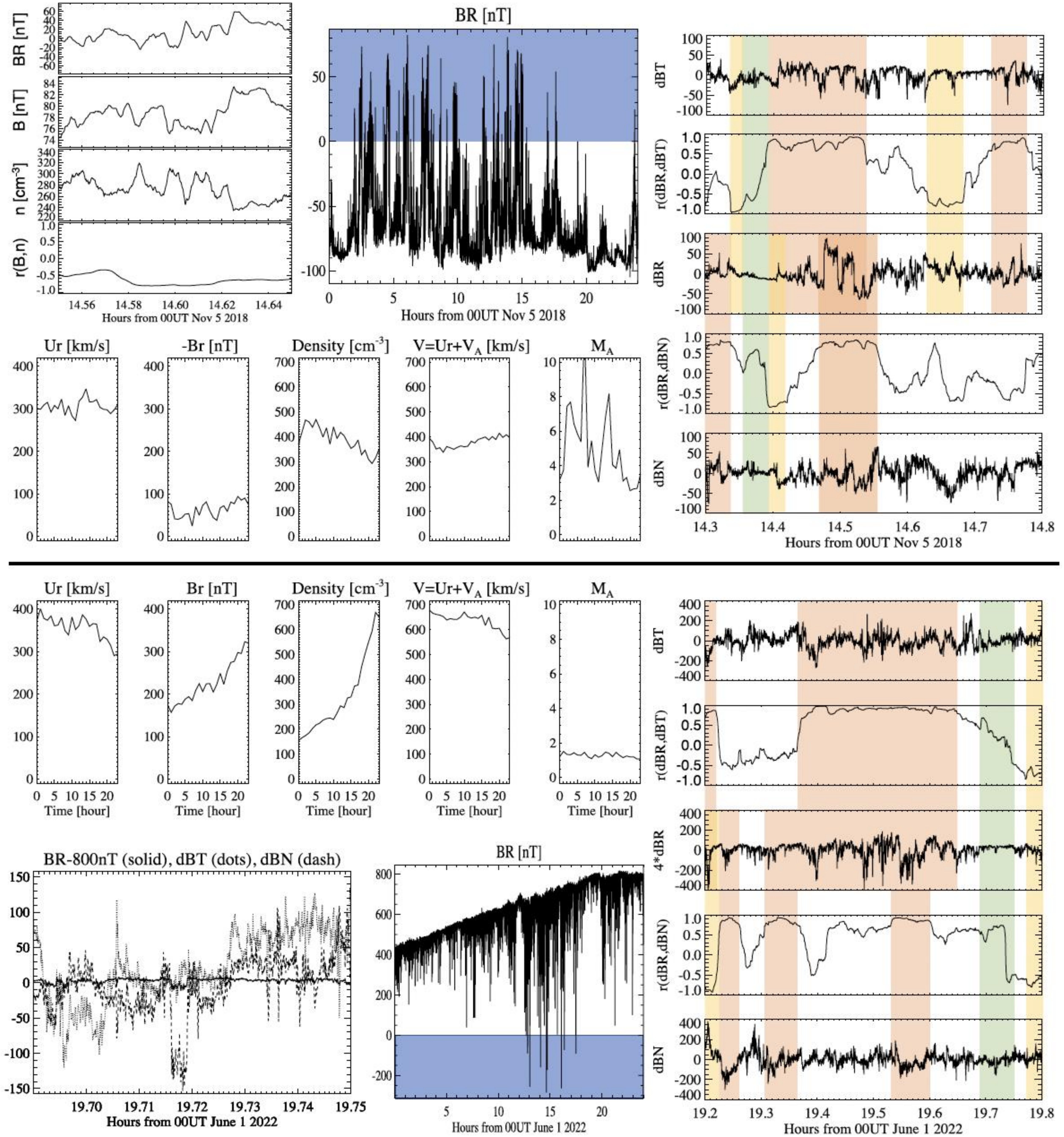
remains the same. Therefore, the choice of $\vec{b}_0$ and $\vec{u}_0$ is actually **arbitrary** so long as $\vec{b}_1 = -\vec{u}_1$.





u_1
b_1
b_0
u_1'
b_1'
b_0'

Switchback Patches


BR [nT]
B [nT]
n [cm^-3]
r(B,n)
Hours from 00UT Nov 5 2018
BR [nT]
Hours from 00UT Nov 5 2018
dBT
r(dBR,dBT)
dBR
r(dBR,dBN)
dBN
Hours from 00UT Nov 5 2018
Ur [km/s]
-Br [nT]
Density [cm^-3]
V=Ur+V_A [km/s]
M_A
Ur [km/s]
Br [nT]
Density [cm^-3]
V=Ur+V_A [km/s]
M_A
Time [hour]
Time [hour]
Time [hour]
Time [hour]
Time [hour]
dBT
r(dBR,dBT)
4*dBR
r(dBR,dBN)
dBN
Hours from 00UT June 1 2022
BR-800nT (solid), dBT (dots), dBN (dash)
Hours from 00UT June 1 2022
BR [nT]
Hours from 00UT June 1 2022

Creating Switchbacks

$$v = u_r + v_A$$
$$= u_r + |B_r|/\sqrt{\mu_0 \rho}$$

$$v(y) = v_0 + v_1 \sin(2\pi y/\lambda_y)$$

$$w = (B_\perp/B_r)\lambda_r/\pi$$

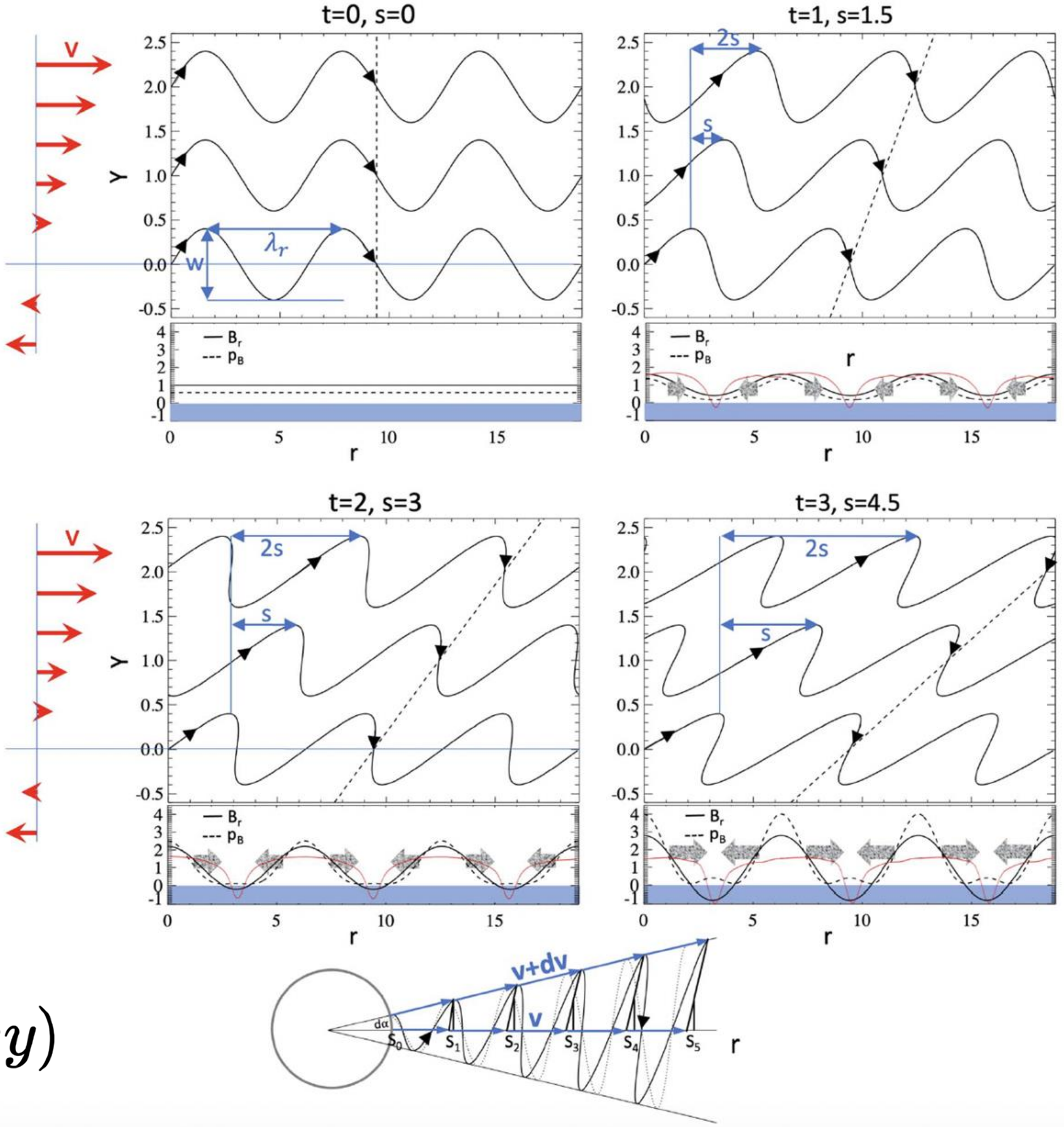
$$\lambda_y \gg w \quad dv/dy \simeq const$$

$$r' = r + sy$$

$$s = t(dv/dy)$$

$$B'_y(r,y) = B_y(r - sy), B'_z(r,y) = B_z(r - sy)$$

$$B'_r(r,y) = B_r + sB_\perp \cos(r - sy)$$



Fast waves, low beta

$$\omega^2 = k^2 \mathrm{V}_\mathrm{a}^2$$

$$\omega^2 = (k_h^2 + k_y^2)\mathrm{V}_\mathrm{a}^2$$

$$\omega^2/V_a^2(y) - k_h^2 = k_y^2$$

$$\omega^2/V_a^2 >> k_h^2$$

Total reflection occurs where

$$\omega^2/V_a^2 - k_h^2 = k_y^2 < 0$$

Alfvén waves in an inhomogeneous medium: gradients along the field

$$z^{\pm} = v \pm b/\sqrt{4\pi\rho}$$

$$\frac{\partial z^{\pm}}{\partial t} \mp V_a \cdot \nabla z^{\pm} \mp z^{\mp} \cdot \nabla V_a \pm \frac{1}{2}(z^{\pm} - z^{\mp})\nabla \cdot V_a = 0$$

Energy flux is conserved along the field

$$\frac{dV_a}{dy} = V_a' = -\frac{1}{2}\frac{V_a}{\rho}\frac{d\rho}{dy} = -\frac{1}{2}\frac{V_a}{\rho}\rho'$$

$$S^{+} - S^{-} = S_{\infty},$$

$$S^{\pm} = \rho V_a |\mathbf{z}^{\pm}|^2/8,$$

Equations become

$$k = \omega/\mathrm{V_a} \qquad k'/k = -\mathrm{V_a'}/\mathrm{V_a}$$

$$\mathrm{z}^{\pm} = \rho^{1/4}\mathrm{z}^{\pm}$$

$$\mathrm{z}^{\pm\prime} \mp ik\mathrm{z}^{\pm} - \frac{1}{2}\frac{k'}{k}\mathrm{z}^{\mp} = 0$$

Propagation vs reflection is determined by

$$\epsilon_a = |k'/2k^2| = |V_a'/2\omega|$$

Never total reflection.

Waves and Turbulence in the Solar Wind

Elssäser variables

$$\mathbf{z}^{\pm} = \delta\mathbf{U} \pm \frac{\delta\mathbf{B}}{\sqrt{4\pi\rho}}$$

Alfvén speed

$$V_a = \frac{B}{\sqrt{4\pi\rho}}$$

convection | reflection &shear | reflection & expansion

$$\frac{\partial\mathbf{z}^{\pm}}{\partial t} + (\mathbf{U} \mp \mathbf{V}_a)\cdot\nabla\mathbf{z}^{\pm} = -\mathbf{z}^{\mp}\cdot\nabla(\mathbf{U} \pm \mathbf{V}_a) + \frac{1}{2}(\mathbf{z}^{-} - \mathbf{z}^{+})\nabla\cdot(\mathbf{V}_a \pm \mathbf{U}/2)$$

$$-\frac{1}{\rho}\nabla P - \mathbf{z}^{\mp}\cdot\nabla\mathbf{z}^{\pm} + < \mathbf{z}^{\mp}\cdot\nabla\mathbf{z}^{\pm} > + \nu^{\pm}\nabla^2\mathbf{z}^{\pm}$$

nonlinearity | dissipation

- ***Both nonlinearities and expansion effects (inhomogeneity) are crucial***
- ***Turbulence in the solar wind is a multi-scale problem: must resolve scales from ~1AU down to at least 10^{-5} AU — incredibly challenging for computation***

O(0) Alfvén wave model

$$\nabla \cdot \vec{V}_g S = 0 \quad \vec{V}_g = \vec{U} + \vec{V}_a$$

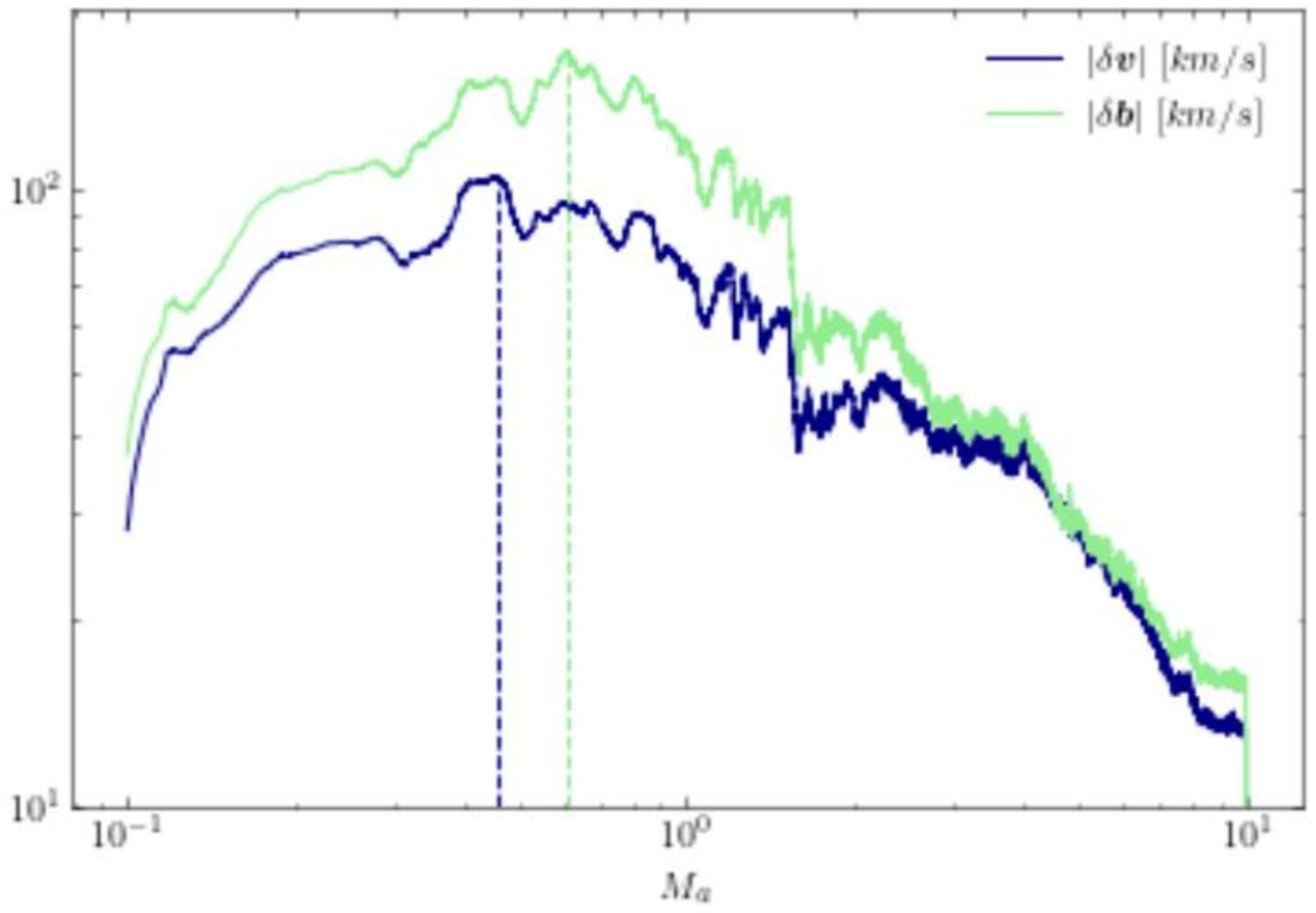
$$S = \frac{1}{2}\rho\frac{|Z^-|^2}{\omega} \qquad \omega = k(V_a) \quad k = \frac{\omega_0}{U + V_a}$$

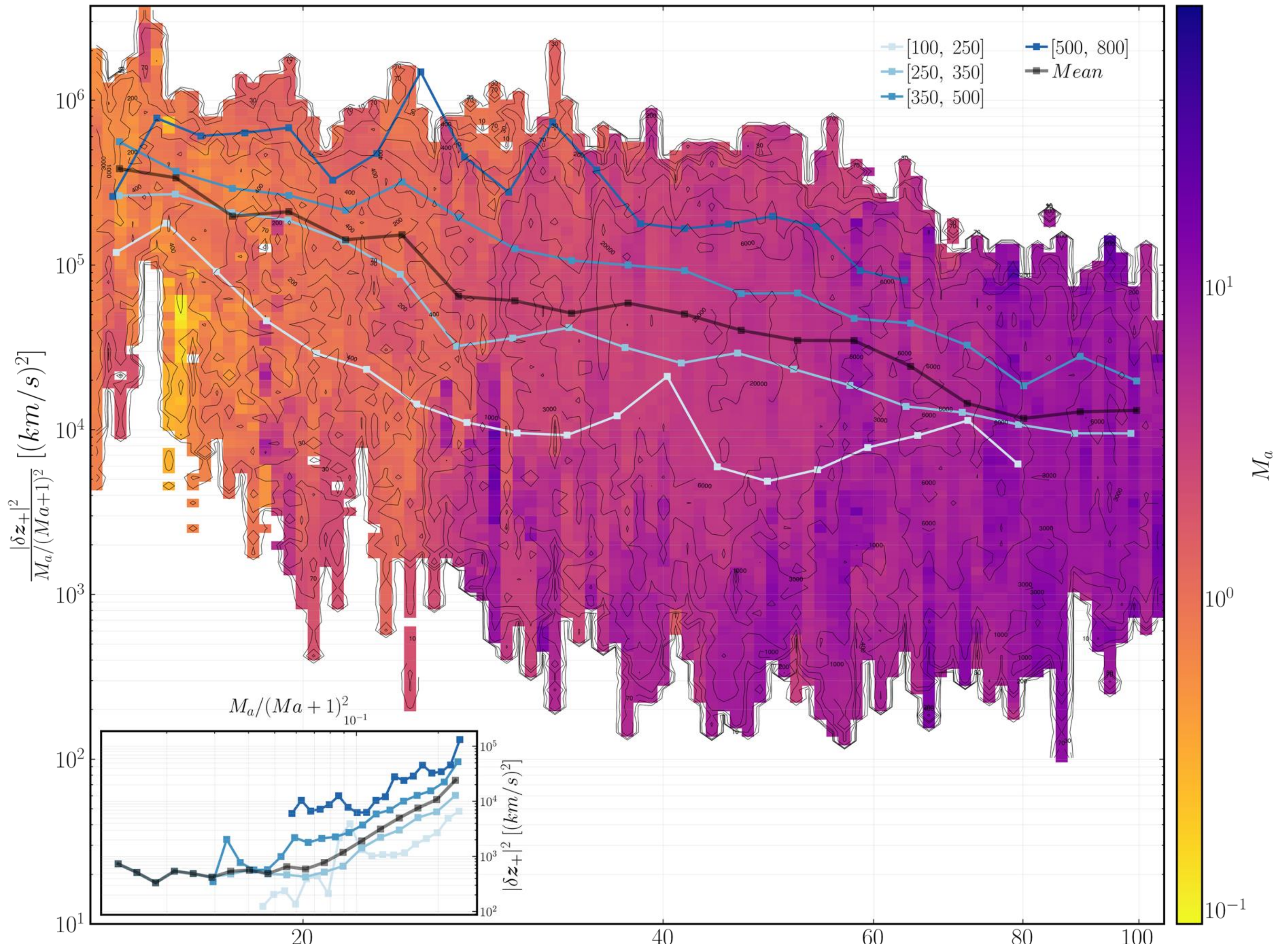
$$\rho U R^2 = \dot{M} \qquad \frac{(U + V_a)^2}{U V_a}|Z^-|^2 = K$$

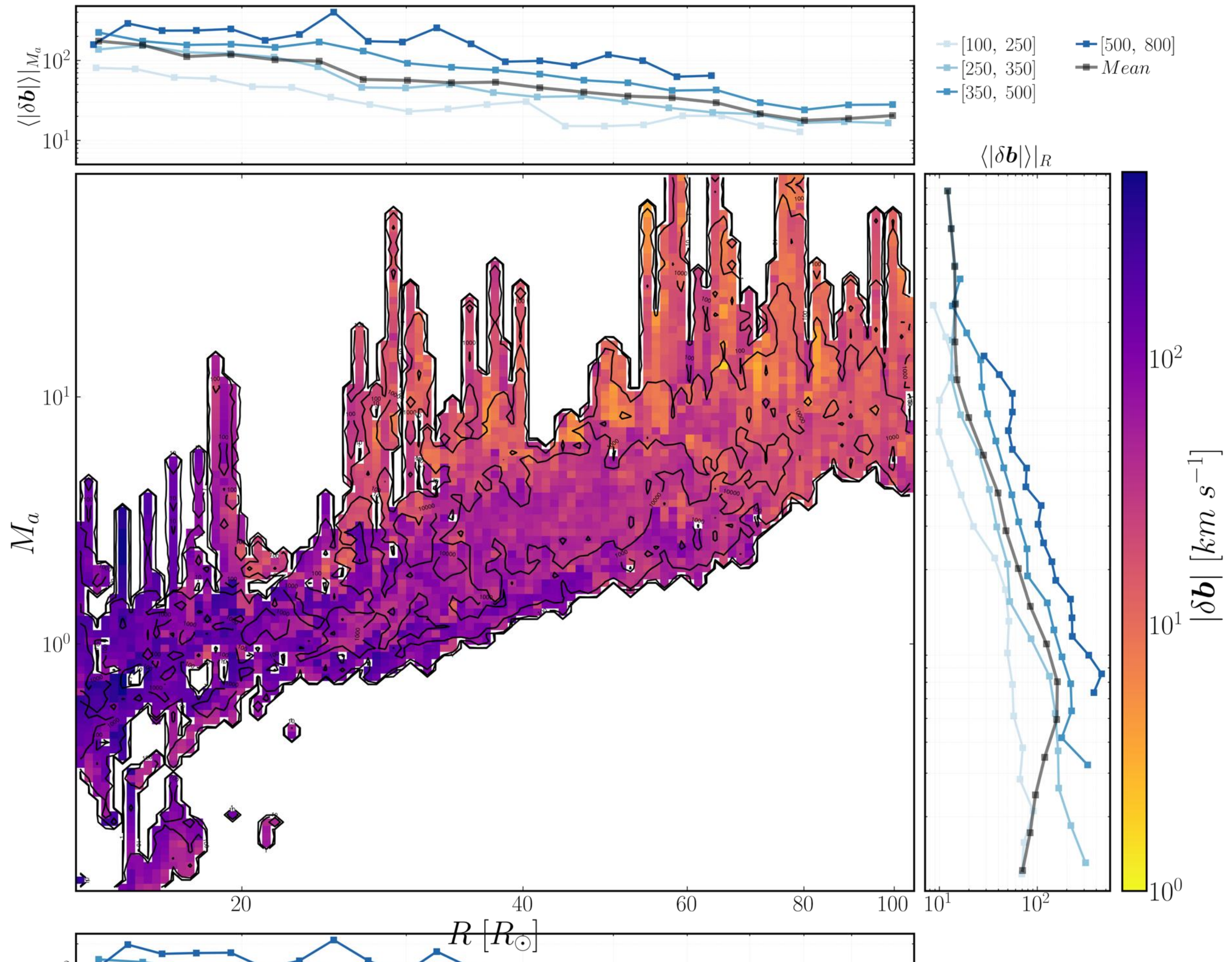
Close to the Sun U<<V_a far from the Sun U>>V_A

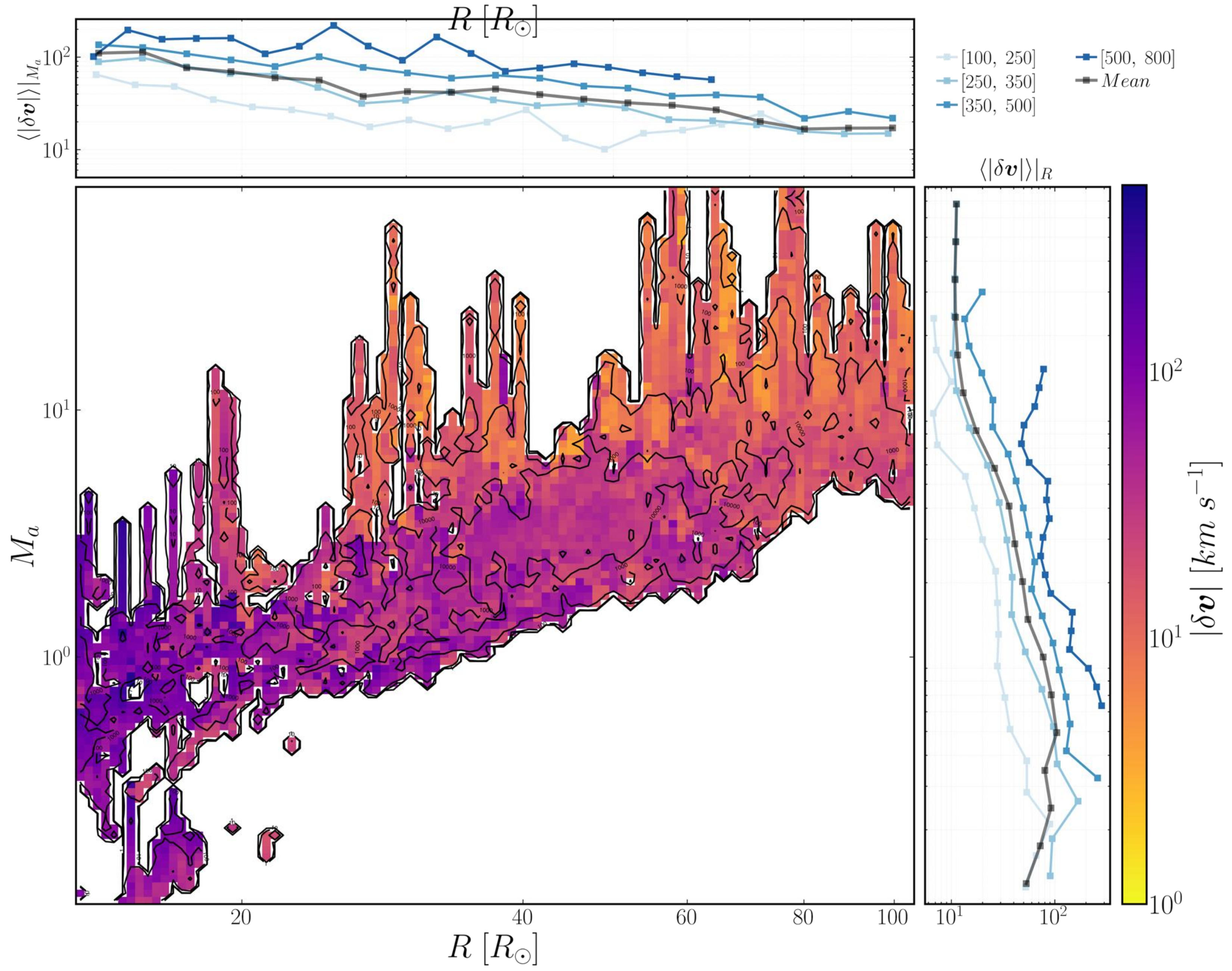
$$\left(\frac{V_a}{U}\right)_\odot |Z^-|^2_\odot \geq \left(\frac{U}{V_a}\right)_W |Z^-|^2_W$$

$$\frac{(U+V_a)^2}{UV_a}|Z^-|^2 = K$$









Magnetic Reconnection and the Tearing Instability

Ohm's law

$$\vec{E} = -\frac{1}{c}\vec{U} \times \vec{B} + \frac{\vec{J}}{\sigma} \qquad \eta = \frac{1}{\sigma} \qquad \vec{J} = \frac{c}{4\pi}(\vec{\nabla} \times \vec{B}) \qquad \eta_m = \frac{\eta c^2}{4\pi}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{U} \times \vec{B} + \vec{\nabla} \times \frac{\eta c^2}{4\pi} \nabla \times \vec{B}$$

Generalized Ohm's law (e- equation of motion)

$$E_i + \frac{1}{c}(\vec{U} \times \vec{B})_i - \frac{J_i}{\sigma} = \frac{m_e}{e^2 n_e}\left[\frac{\partial J_i}{\partial t} + \frac{\partial}{\partial x_k}\Big(J_i U_k + J_k U_i\Big)\right] +$$

$$+ \frac{1}{e n_e c}\Big(\vec{J} \times \vec{B}\Big)_i - \frac{1}{e n_e}\frac{\partial P_{ik}^{(e)}}{\partial x_k}$$

$$E_i + \frac{1}{c}(\vec{U} \times \vec{B})_i - \frac{J_i}{\sigma} = \frac{m_e}{e^2 n_e}\left[\frac{\partial J_i}{\partial t} + \frac{\partial}{\partial x_k}\Big(J_i U_k + J_k U_i\Big)\right] +$$

$$+ \frac{1}{e n_e c}\Big(\vec{J} \times \vec{B}\Big)_i - \frac{1}{e n_e}\frac{\partial P_{ik}^{(e)}}{\partial x_k}$$

Let us now carry out a dimensional analysis by dividing all terms in color with the first term (a generic field) The frequency is defined as the inverse of the characteristic time scale and we define the sound speed

$$\omega \simeq \tau^{-1} \quad , \quad c_s \simeq \Big(P/\rho\Big)^{\frac{1}{2}}$$

$$1:1 : \Big(\omega/\omega_{pe}\Big)\Big(\nu_{ei}/\omega_{pe}\Big)\Big(c/\mathcal{U}\Big)^2 : \Big(\omega/\omega_{pe}\Big)\Big(\omega_{ce}/\omega_{pe}\Big)\Big(c/\mathcal{U}\Big)^2 :$$

$$\Big(\omega/\omega_{pe}\Big)^2\Big(c/\mathcal{U}\Big)^2 : \Big(\omega/\omega_{cp}\Big)^2\Big(c_s/\mathcal{U}\Big)^2 :$$

$$\omega_{pe} \qquad \omega_{ce}, \omega_{cp} \qquad \sigma = \frac{e^2 n_e}{m_e \nu_{ep}}$$

The electron plasma frequency, electron and proton cyclotron frequency and the collsion frequencies also appear. It follows that to neglect the terms in color boxes we require, for the inertial term in green:

$$\left(\omega/\omega_{pe}\right) \ll \mathcal{U}/c$$

For the hall term proportional to

$$\vec{J} \times \vec{B} \qquad \left(\omega\omega_{ce}/\omega_{pe}^2\right) \ll \left(\mathcal{U}/c\right)^2$$

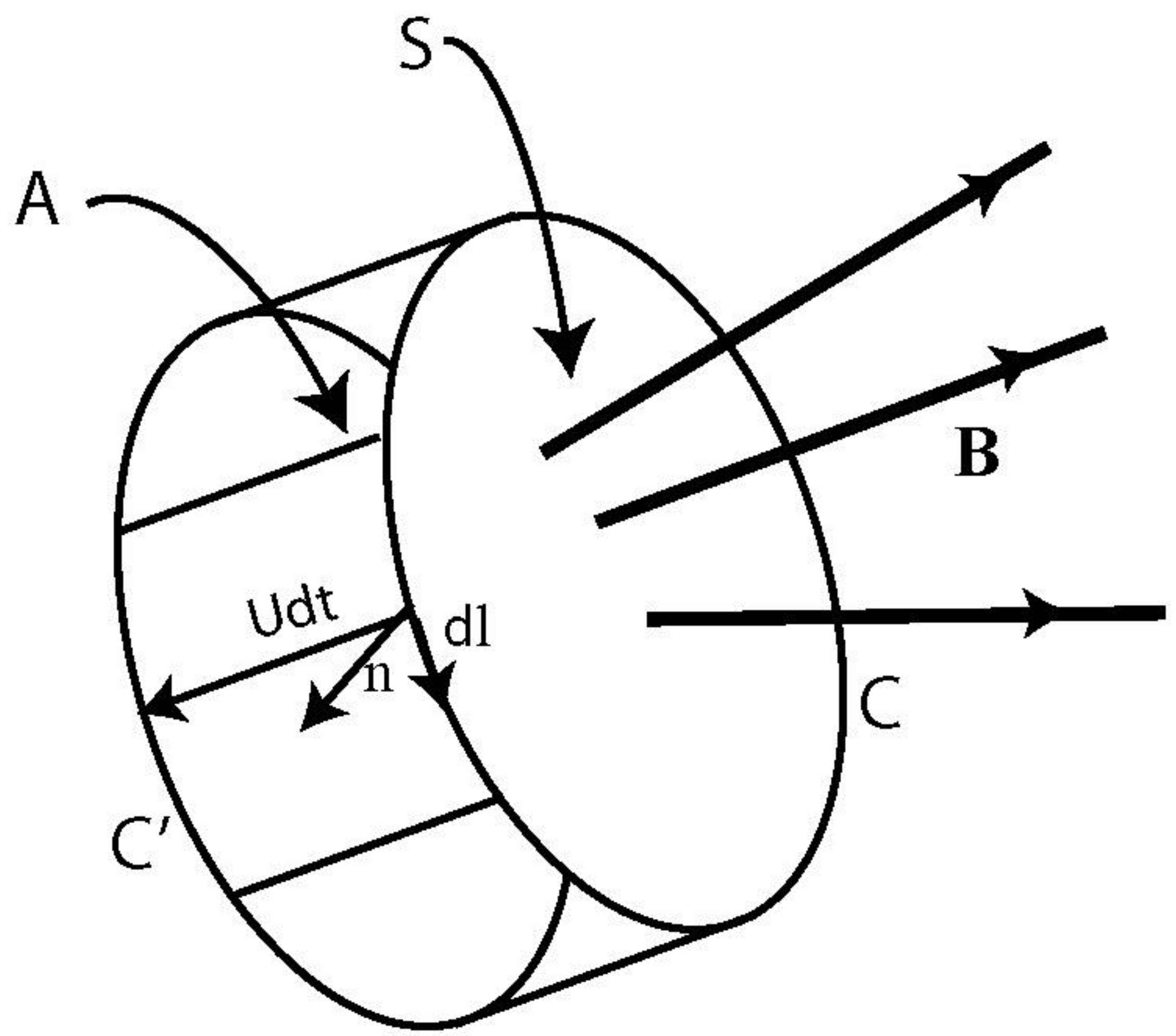
This may also be written differently, taking into account that the Hall term may also be written in terms of the proton cyclotron frequency and the Alfven

$$[\frac{1}{en_e c}\left(\vec{J} \times \vec{B}\right)_i]\frac{c}{\mathcal{U}\mathcal{B}} = \omega\frac{cm_p}{e\mathcal{B}}\frac{\mathcal{B}^2}{n_e m_p}\frac{1}{\mathcal{U}^2} = \frac{\omega}{\omega_{ci}}(\frac{V_a}{U})^2 \rightarrow \quad \frac{\omega}{\omega_{ci}} \ll (\frac{U}{V_a})^2$$

While the electron pressure gradient term may be eliminated when

$$\left(\omega/\omega_{cp}\right) \ll \left(\mathcal{U}/c\right)^2$$

Alfvén Theorem: magnetic flux through a closed line which moves with the fluid is constant in time.



$$\Phi = \int \vec{B} \cdot \mathrm{d}\vec{S} \qquad \Delta\Phi = \int_{S'} \vec{B}' \cdot \mathrm{d}\vec{S}' - \int_S \vec{B} \cdot \mathrm{d}\vec{S}$$

$$\vec{B}' = \vec{B}(t + \Delta t, \vec{r}) = \frac{\partial \vec{B}}{\partial t}\Delta t + \vec{B}(t, \vec{r})$$

$$\Delta\Phi = \int_S (\vec{B}' - \vec{B}) \cdot \mathrm{d}\vec{S} + \vec{B} \cdot \vec{n}A \qquad \vec{n}A = -\mathrm{d}\vec{l} \times \vec{U}\Delta t$$

$$\vec{B} \cdot \vec{n}A = -\vec{B} \cdot \mathrm{d}\vec{l} \times \vec{U}\Delta t = -\mathrm{d}\vec{l} \cdot \vec{U} \times \vec{B}\Delta t$$

$$\Delta\Phi = \int_S (\vec{B}' - \vec{B}) \cdot \mathrm{d}\vec{S} - \int_C (\vec{U} \times \vec{B}) \cdot \mathrm{d}\vec{l}\Delta t$$

$$\frac{\Delta\Phi}{\Delta t} = \int_S \left(\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{U} \times \vec{B}) \right) \cdot \mathrm{d}\vec{S} = 0$$

Magnetic Helicity: In ideal MHD the helicity density integral along any closed field line is conserved.

$$\mathcal{H} = \int_V \vec{A} \cdot \vec{B}\, d^3 r$$

$$\frac{\partial \vec{A}}{\partial t} = \vec{U} \times (\vec{\nabla} \times \vec{A}) + \nabla \Phi$$

$$\frac{d\mathcal{H}}{dt} = \int_V \left(\frac{\partial \vec{A}}{\partial t} \cdot \vec{B} + \frac{\partial \vec{B}}{\partial t} \cdot \vec{A} \right) + \nabla \cdot \vec{B} \Phi \, d^3 r.$$

$$\frac{d\mathcal{H}}{dt} = \int_V \left[\vec{\nabla} \cdot \left(\frac{\partial \vec{A}}{\partial t} \times \vec{A} \right) + 2\, \frac{\partial \vec{A}}{\partial t} \cdot (\vec{\nabla} \times \vec{A}) \right] d^3 r$$

Minimum energy at constant helicity: Force-free

$$W_B - \lambda \mathcal{H} \qquad \delta\vec{B} = \vec{\nabla} \times \delta\vec{A}$$

$$\begin{aligned}
0 = \delta(W_B - \alpha_0/8\pi\mathcal{H}) &= \frac{1}{8\pi}\delta\int_V (B^2 - \alpha_0 \vec{A}\cdot\vec{B})\, d^3r \\
&= \frac{1}{8\pi}\int_V \left[\vec{B}\cdot\delta\vec{B} - \frac{\alpha_0}{2}(\vec{A}\cdot\delta\vec{B} + \vec{B}\cdot\delta\vec{A})\right] d^3r \\
&= \frac{1}{8\pi}\int_V \left[(\vec{B} - \frac{\alpha_0}{2}\vec{A})\cdot\delta\vec{B} - \frac{\alpha_0}{2}\vec{B}\cdot\delta\vec{A}\right] d^3r
\end{aligned}$$

$$\int_V \vec{\nabla}\cdot\left([\vec{B} - (\alpha_0/2)\vec{A}] \times \delta\vec{A}\right) d^3r + \int_V [\vec{\nabla}\times\vec{B} - \alpha_0\vec{B}]\cdot\delta\vec{A}\, d^3r = 0$$

If resistivity is included, one can show that the Total Helicity must be dissipated at a rate which is much lower than the total energy. From definitions:

$$H = \int_V \mathbf{A} \cdot \mathbf{B}\, d^3r \qquad \frac{dH}{dt} = -2 \int_V \eta c \mathbf{J} \cdot \mathbf{B}\, d^3r$$

$$E_M = \int_V \frac{\mathbf{B}^2}{8\pi} d^3r \qquad \frac{dE_M}{dt} = -\int_V \eta \mathbf{J}^2$$

Apply now the Schwarz inequality $\left(\int_V \mathbf{J} \cdot \mathbf{B}\, d^3r\right)^2 < \int_V \mathbf{J}^2 d^3r \int_V \mathbf{B}^2 d^3r$

To obtain

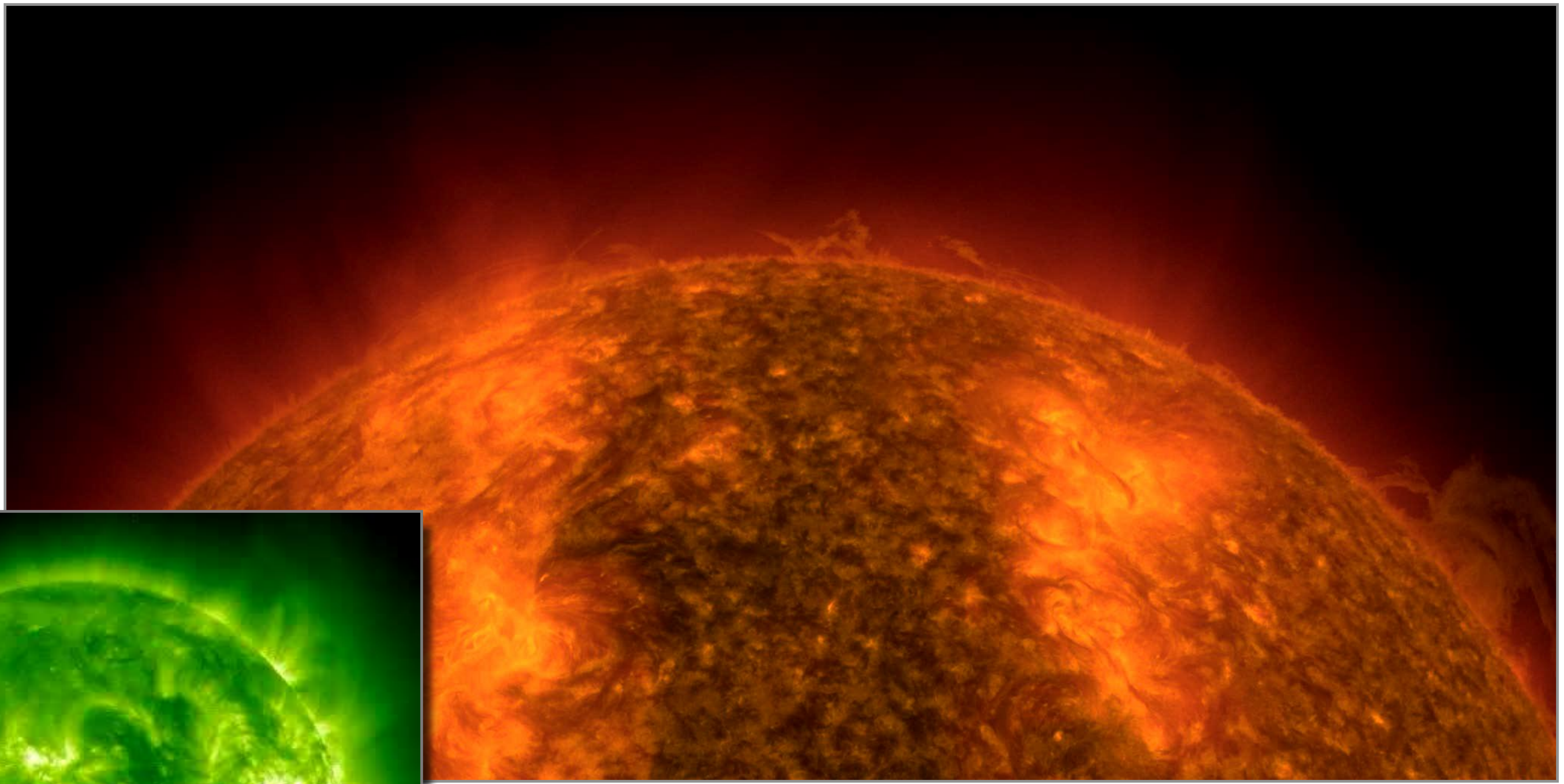
$$\left|\frac{dH}{dt}\right| < k\sqrt{\eta}\sqrt{\left|\frac{dE_M}{dt}\right| E_M} \qquad k = 2c\sqrt{8\pi}$$

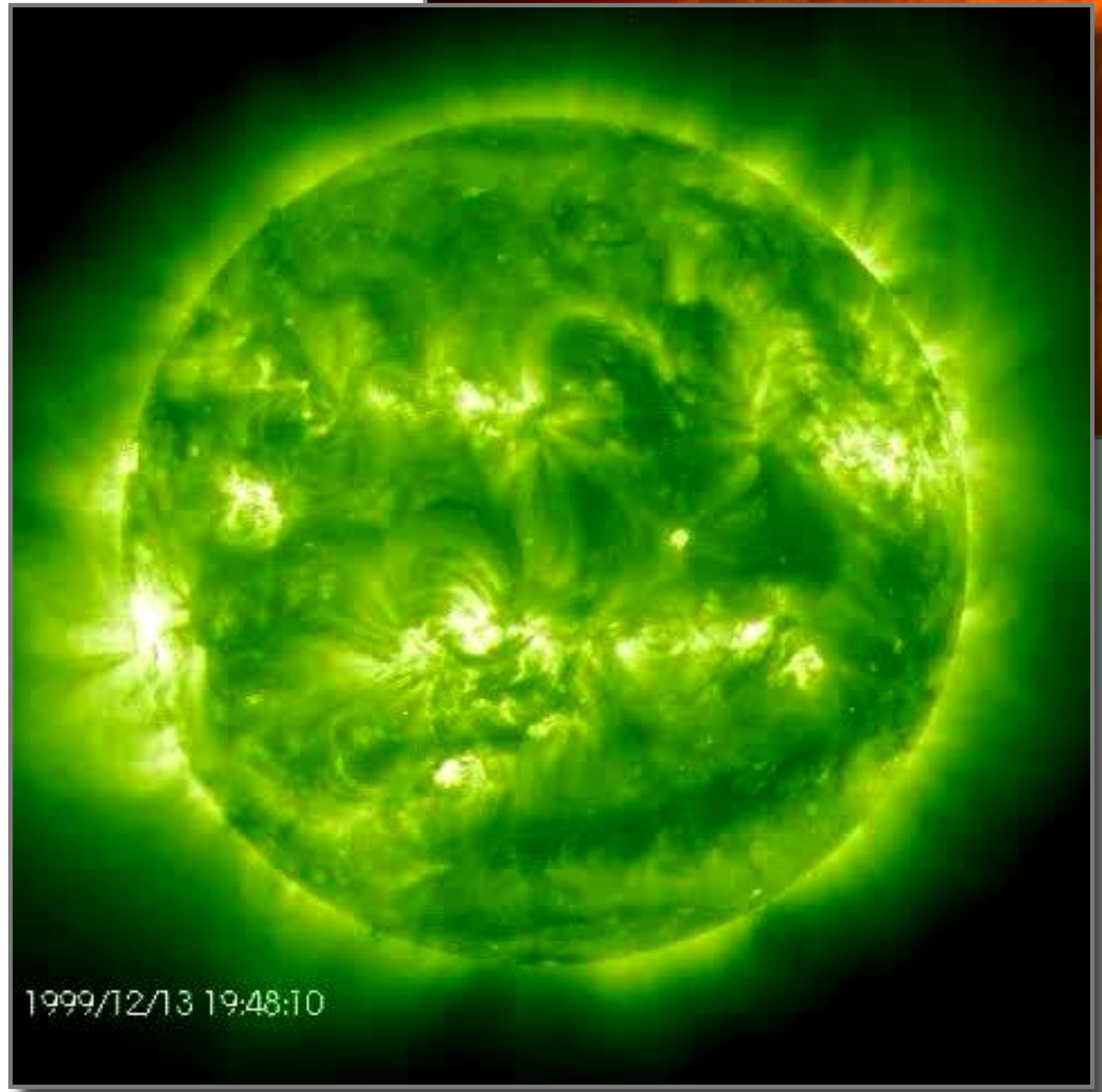
This implies that in the presence of turbulence or reconnection (finite energy dissipation with vanishingly small $\eta \rightarrow 0$ ***total helicity CAN NOT be dissipated.***

Alfvén Theorem: magnetic flux through a closed line which moves with the fluid is constant in time. Resistivity breaks the Alfvén theorem leading to possible energy dissipation and relaxation.

Energy minimum (Woltjer) Relaxation (Taylor) -> Force-free states
Astrophysical Applications: Coronal heating, Magnetic structure of the solar corona. But how?

Solar Corona: from Flares and CMEs to Heating



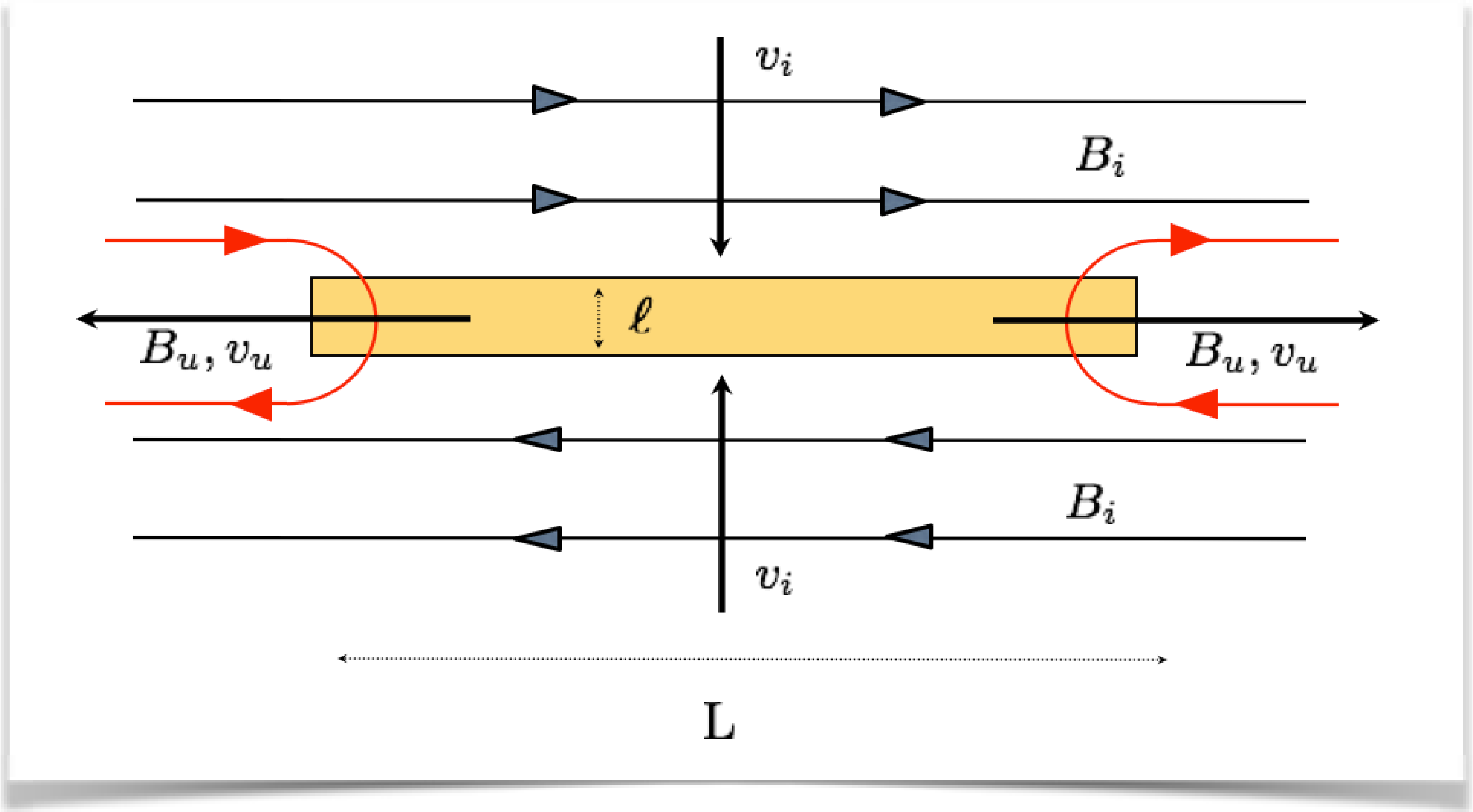


Flare: Rapid energy release Coronal Heating: Integrated Rapid Energy Release (1 Large Flare = 100 hrs of heating on 1 Active region)

$E \simeq 10^{32}\text{erg} \quad \mathcal{L} \sim 10^9 - 10^{10}\,\text{cm} \quad \tau \leq 1600s$

Original Stationary Reconnection Scenario

"Stationary, *driven*" Sweet-Parker reconnection of a current sheet (and its extensions, e.g., the Petschek model)



$$\ell / L = S^{-1/2}$$

$$S = \frac{L V_A}{\eta_m}$$

$$R := \frac{v_i}{v_A} = \left(\frac{\eta}{L v_a}\right)^{1/2} = S^{-1/2}$$

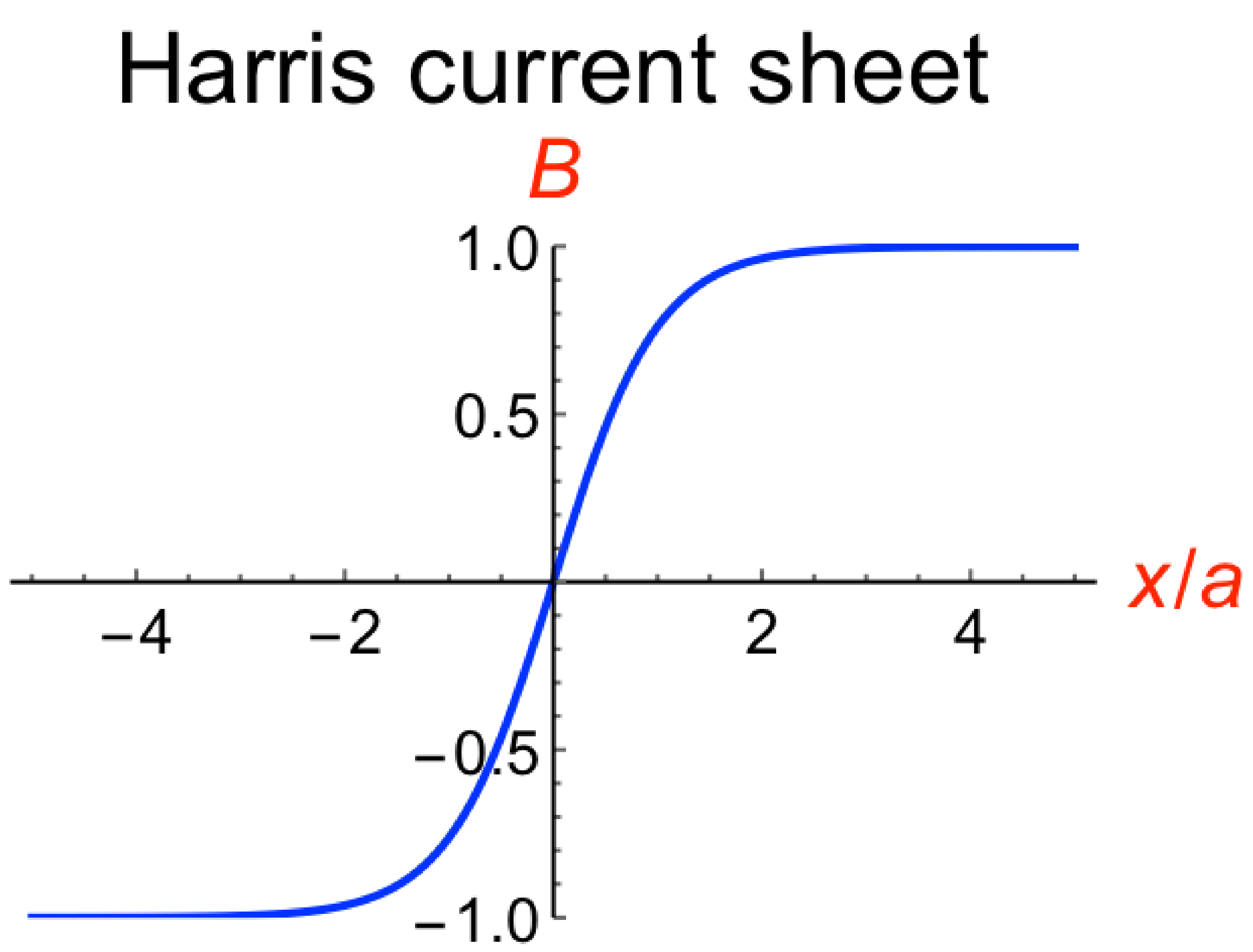
SLOW

Typical timescale of energy release is 3-30 years!

"The observational and theoretical difficulties with the hypothesis of magnetic-field line annihilation suggest that other alternatives for the flare must be explored." **E. Parker, 1963**

- ***"Spontaneous" reconnection as the outcome of an internal instability, namely, the tearing mode instability [Furth et al. 1963] on the HARRIS SHEET***

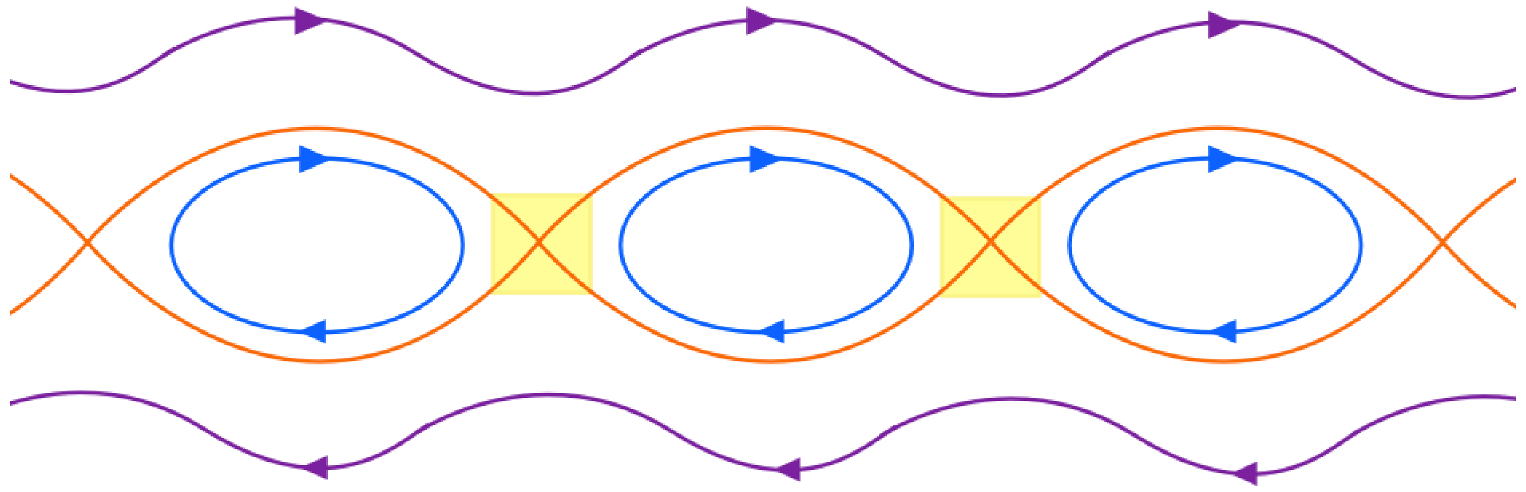
$$\begin{cases} \vec{B}_0(y) = B_0 tanh\left(\frac{y}{a}\right)\hat{i} \\ \vec{B}_0(y) = B_0 tanh\left(\frac{y}{a}\right)\hat{i} + B_0 sech\left(\frac{y}{a}\right)\hat{k} \end{cases}$$



Normalization

$$v_A := \frac{B_0}{\sqrt{4\pi\rho_0}} \qquad \tau_A := \frac{a}{v_A} \qquad S := \frac{a}{v_A \eta}$$

The Tearing Instability: Linearized Equations



$$\vec{v}_1 = v_x(x,y,t)\hat{i} + v_y(x,y,t)\hat{j} \qquad \vec{B}_1 = b_x(x,y,t)\hat{i} + b_y(x,y,t)\hat{j}$$

Perturbations:

Expand in a Fourier series:

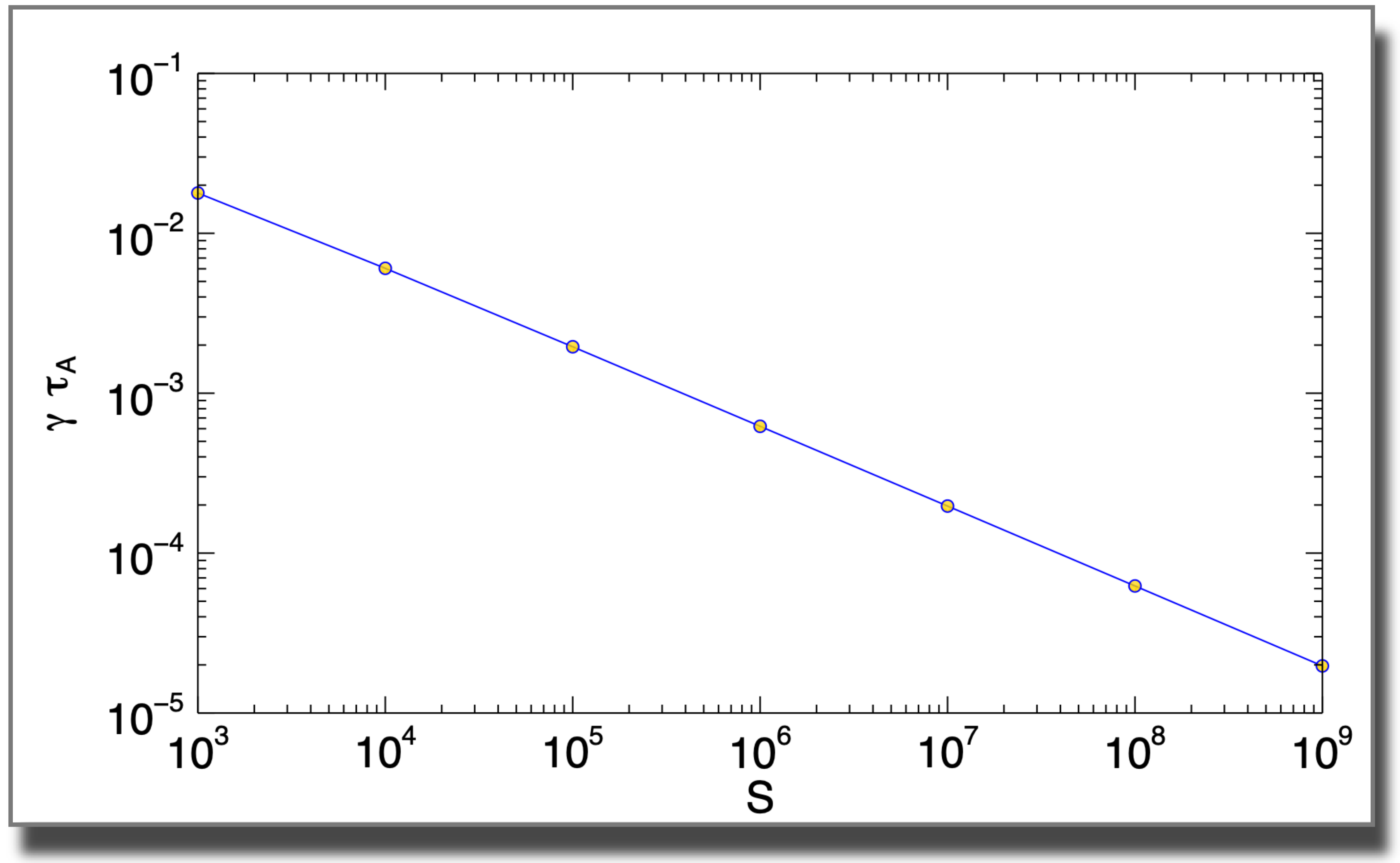
$$f(x,y,t) = f(y)\, e^{ikx} e^{\gamma t}$$

$$b, v \to 0,\ y \to \pm\infty$$

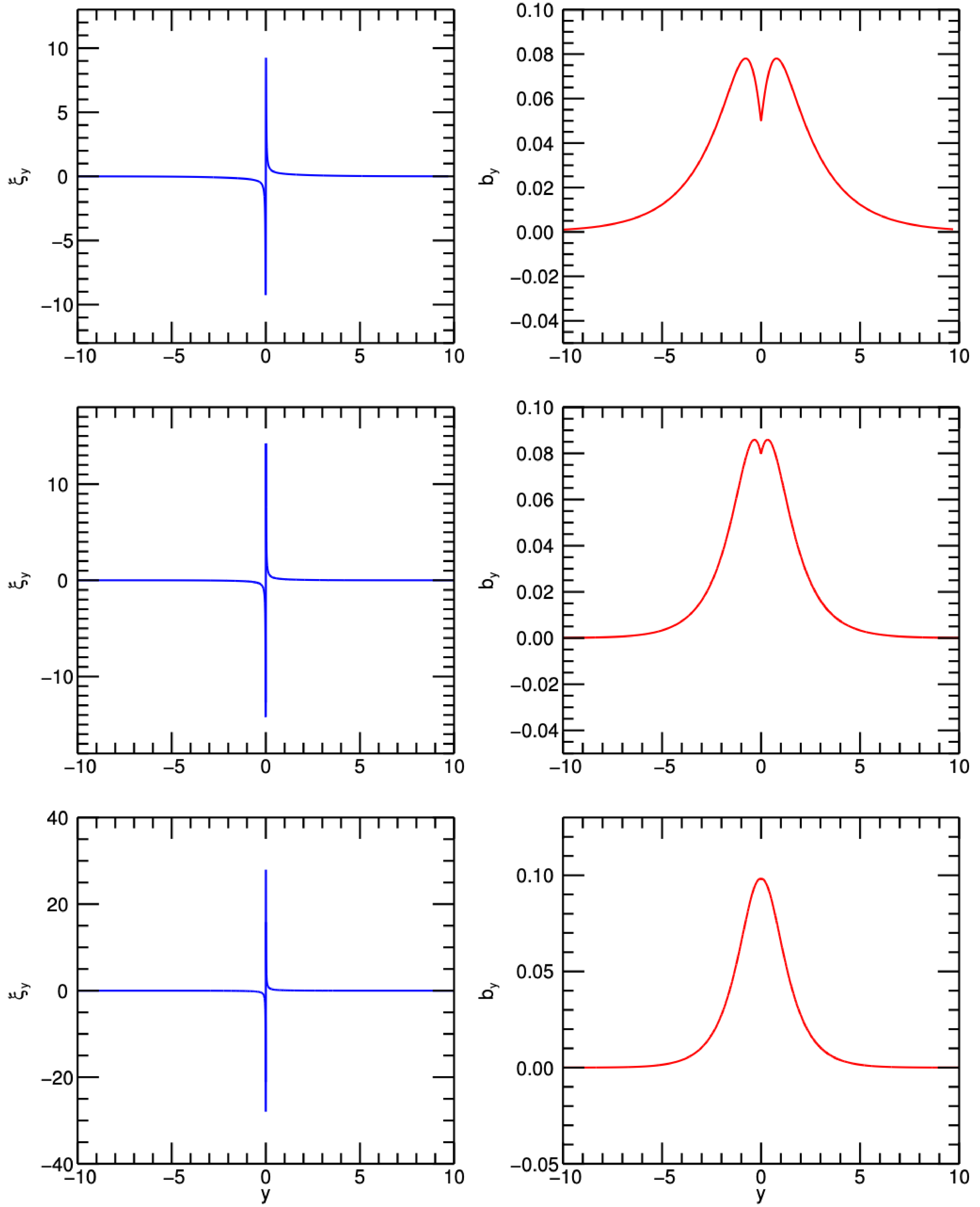
$$\begin{cases} \gamma(v'' - k^2 v) = \dfrac{ik}{4\pi\rho_0}\left[B_{0x}\left(b'' - k^2 b\right) - B''_{0x} b\right] \\ \gamma b = ikB_{0x} v + \eta\left(b'' - k^2 b\right) \end{cases}$$

$$\begin{cases} ikb_x(y) + b'_y(y) = 0 \\ ikv_x(y) + v'_y(y) = 0 \end{cases}$$

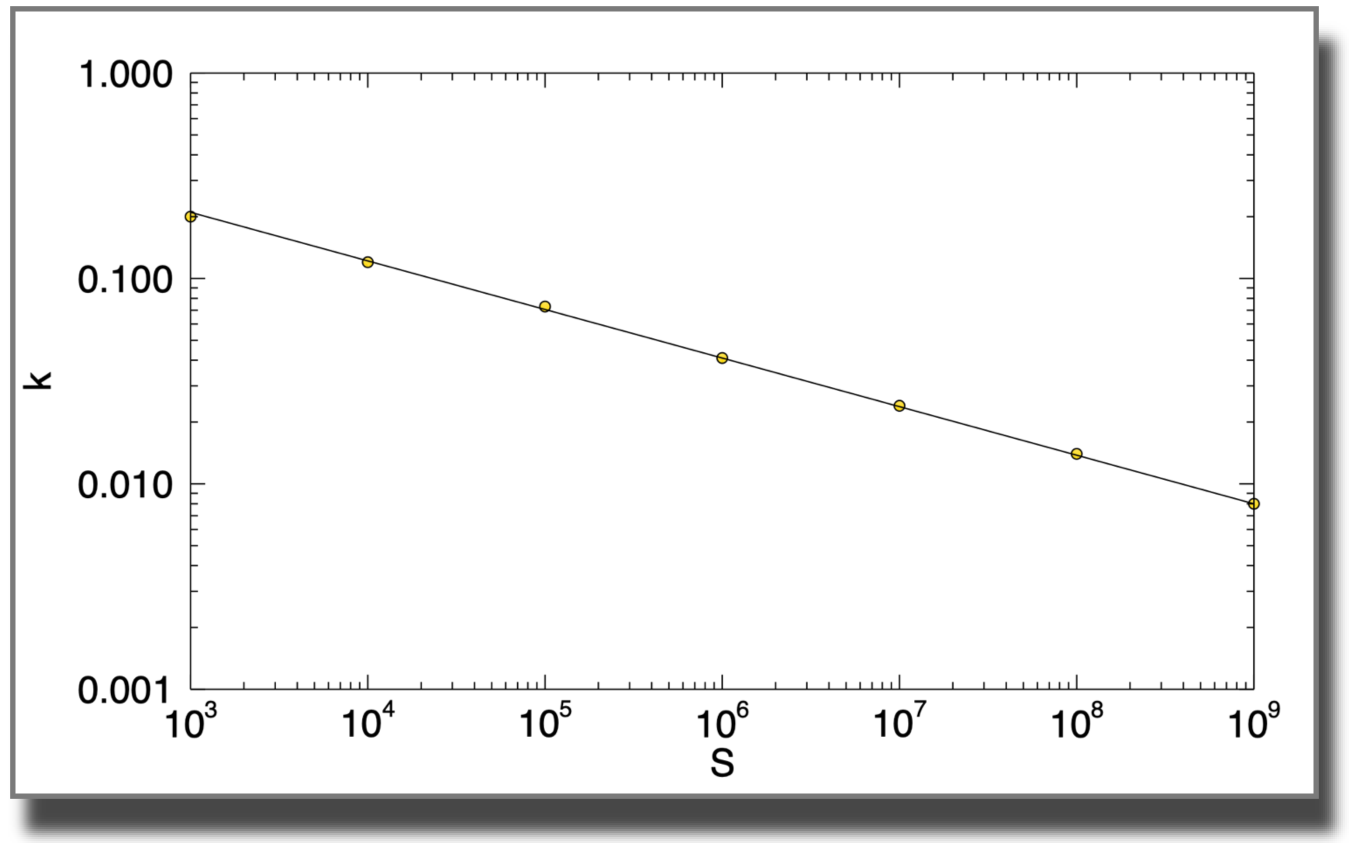
$$\Delta' = a\,\frac{b'(\delta/2) - b'(-\delta/2)}{b(\delta/2)}$$



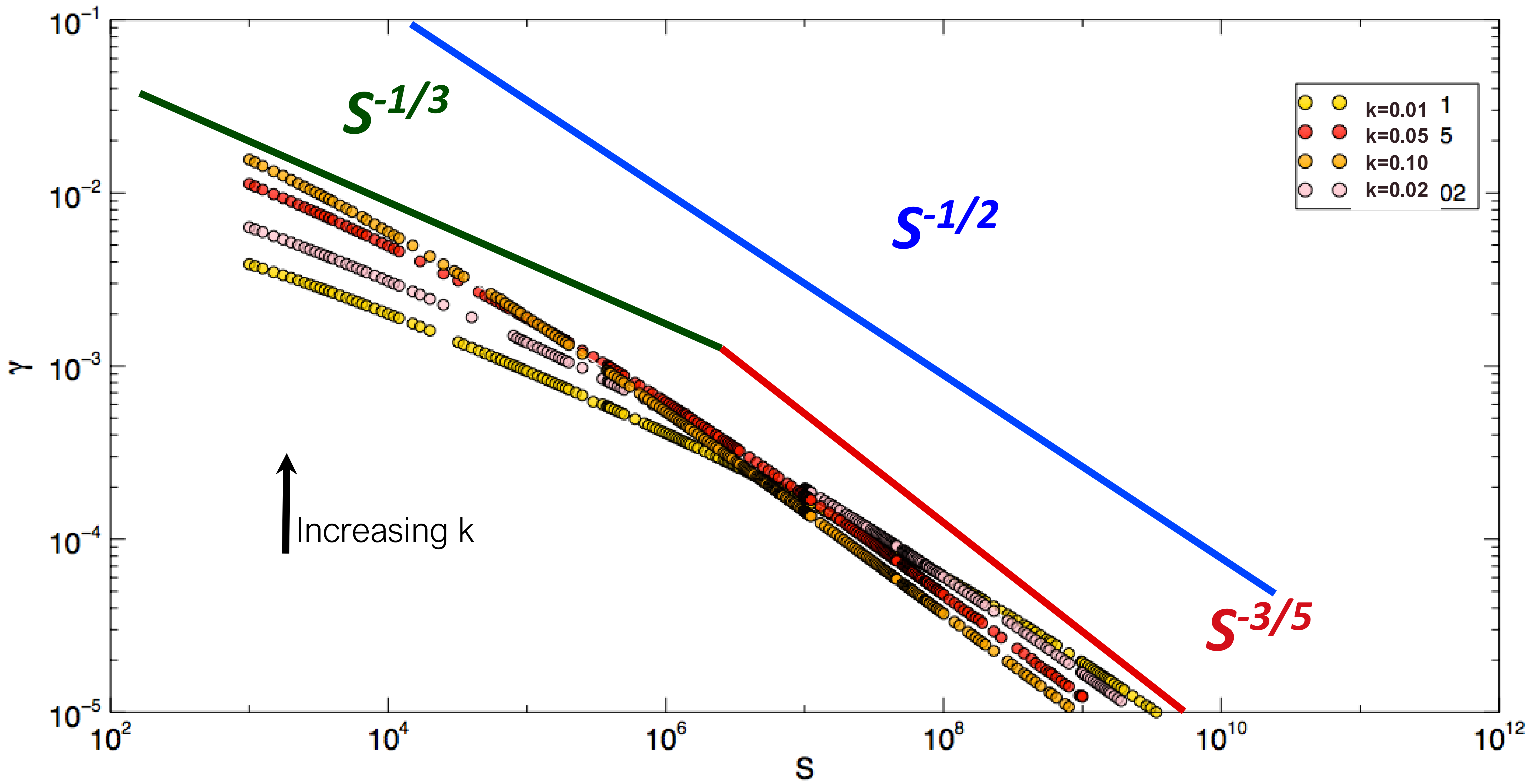
$$\gamma_{max}\tau_A \simeq S^{-1/2}$$



$$k_{max}a \simeq S^{-1/4}$$



Tearing Mode Instability SUMMARY



Three regimes:

SLOW

$$\gamma_m \tau_A \propto S^{-1/2}$$

Fastest Growing Mode

$$\gamma \tau_A \propto S^{-1/3}$$

large Δ'

$$\gamma \tau_A \propto S^{-3/5}$$

small Δ'

Tearing Mode Instability : old and new

Magnetic Reconnection in Plasmas

Dieter Biskamp
Max-Planck-Institute for Plasma Physics, Garching

Earth Planets Space, **53**, 473–482, 2001

Plasmoid-induced-reconnection and fractal reconnection

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As a key to undertanding the basic mechanism for fast reconnection in solar flares, *plasmoid-induced-reconnection* and *fractal reconnection* are proposed and examined. We first briefly summarize recent solar observations that give us hints on the role of plasmoid (flux rope) ejections in flare energy release. We then discuss the plasmoid-induced-reconnection model, which is an extention of the classical two-ribbon-flare model which we refer to as the CSHKP model. An essential ingredient of the new model is the formation and ejection of a plasmoid which play an essential role in the storage of magnetic energy (by inhibiting reconnection) and the induction of a strong inflow into reconnection region. Using a simple analytical model, we show that the plasmoid ejection and acceleration are closely coupled with the reconnection process, leading to a *nonlinear instability* for the whole dynamics that determines the macroscopic reconnection rate uniquely. Next we show that the current sheet tends to have a *fractal structure* via the following process path: tearing $\Rightarrow$ sheet thinning $\Rightarrow$ Sweet-Parker sheet $\Rightarrow$ secondary tearing $\Rightarrow$ further sheet thinning $\Rightarrow \cdots$. These processes occur repeatedly at smaller scales until a microscopic plasma scale (either the ion Larmor radius or the ion inertial length) is reached where anomalous resistivity or collisionless reconnection can occur. The current sheet eventually has a fractal structure with many plasmoids (magnetic islands) of different sizes. When these plasmoids are ejected out of the current sheets, fast reconnection occurs at various different scales in a highly time dependent manner. Finally, a scenario is presented for fast reconnection in the solar corona on the basis of above *plasmoid-induced-reconnection in a fractal current sheet*.

The "plasmoid chain instability" of Sweet-Parker current sheets

- Biskamp (1993) first showed that thin Sweet-Parker current sheets are Tearing unstable for an aspect ratio $a/L<10^{-4}$ (Sonnerup and Sakai 1981)
- With a proper renormalization of Alfvén time and Lundquist # to the macroscopic length L, the maximum growth rate scales as $S^{1/4}$ [Loureiro et al., 2007+]

Alfvén time and Lundquist #: $\tau_A^* = L/v_A, \quad (S^*) = L/(v_A\eta)$

$$\gamma\tau_A := \gamma\tau_A^* a/L \sim S^{-1/2} := (S^*)^{-1/2}(a/L)^{-1/2}$$

Sweet-Parker: $a/L = (S^*)^{-1/2}$

$$\boxed{\gamma\tau_A^* \sim (S^*)^{1/4}}$$

PARADOX !

In nature, Sweet-Parker current sheets cannot be formed in the first place!

The Astrophysical Journal Letters, 780:L19 (4pp), 2014 January 10 doi:10.1088/2041-8205/780/2/L19

RECONNECTION OF QUASI-SINGULAR CURRENT SHEETS: THE "IDEAL" TEARING MODE

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ABSTRACT

A strong indication that fast reconnection regimes exist within resistive magnetohydrodynamics was given by the proof that the Sweet–Parker current sheet, maintained by a flow field with an appropriate inflow–outflow structure, could be unstable to a reconnecting instability which grows without bound as the Lundquist number, S, tends to infinity. The requirement of a minimum value for S in order for the plasmoid instability to kick in does little to resolve the paradoxical nature of the result. Here we argue against the realizability of Sweet–Parker current sheets in astrophysical plasmas with very large S by showing that an "ideal" tearing mode takes over before current sheets reach such a thickness. While the Sweet–Parker current sheet thickness scales as $\sim S^{-1/2}$, the tearing mode becomes effectively ideal when a current sheet collapses to a thickness of the order of $\sim S^{-1/3}$, up to 100 times thicker than $S^{-1/2}$, when (as happens in many astrophysical environments) S is as large as 10^{12}. Such a sheet, while still diffusing over a very long time, is unstable to a tearing mode with multiple x-points: here we detail the characteristics of the instability and discuss how it may help solve the flare trigger problem and effectively initiate the turbulent disruption of the sheet.

Key words: magnetic reconnection – magnetohydrodynamics (MHD) – Sun: corona

“Ideal” Tearing mode in Thin Sheets

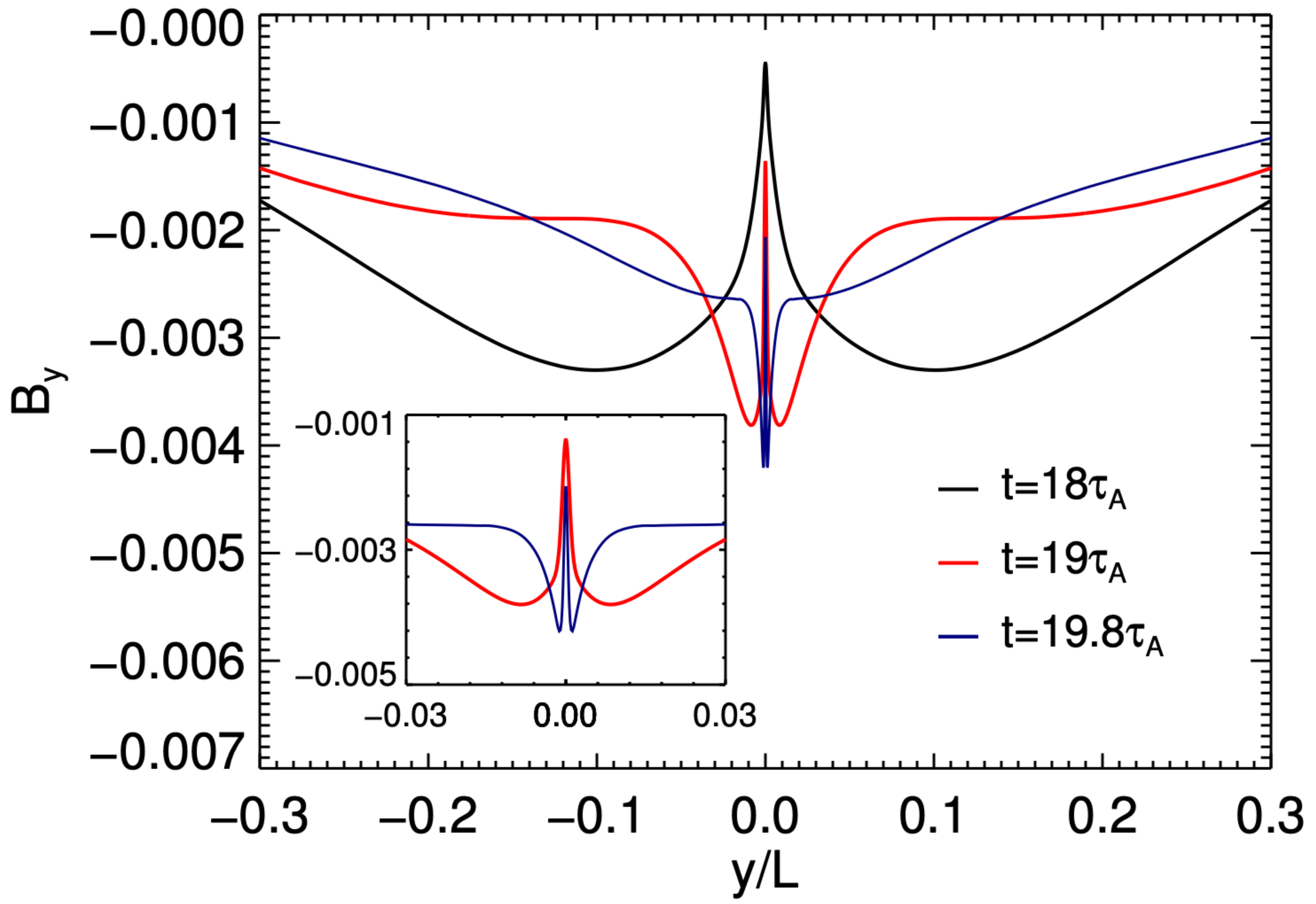
- Sweet-Parker current sheets have a growth rate which diverges in the ideal limit S>>1
- CONJECTURE: there is a critical aspect ratio L/a at which the growth rate does not depend on S. As such, that aspect ratio provides an upper limit to current sheets that can be formed in nature [Pucci and Velli APJL 2014]

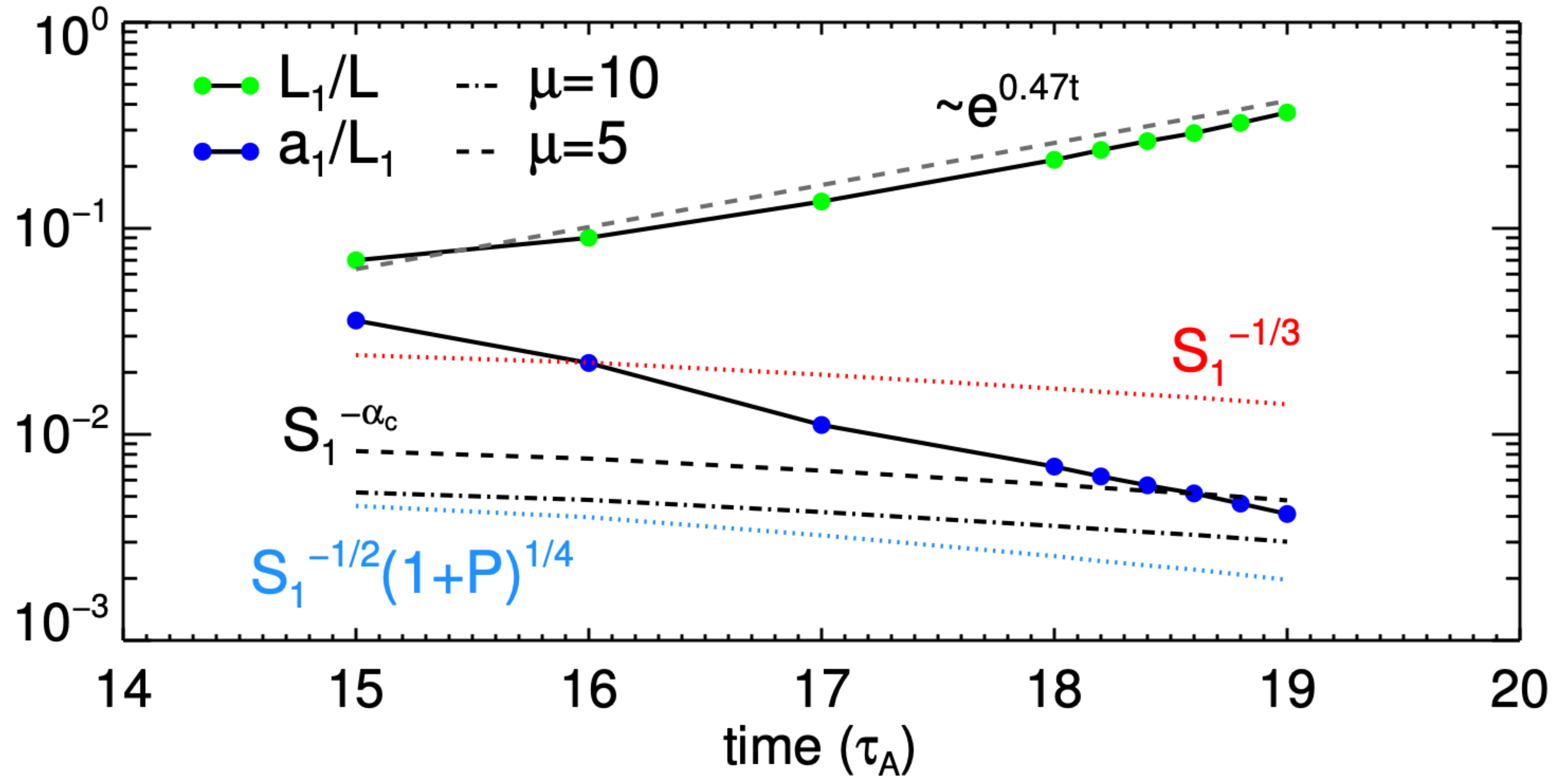
width: a, length: L

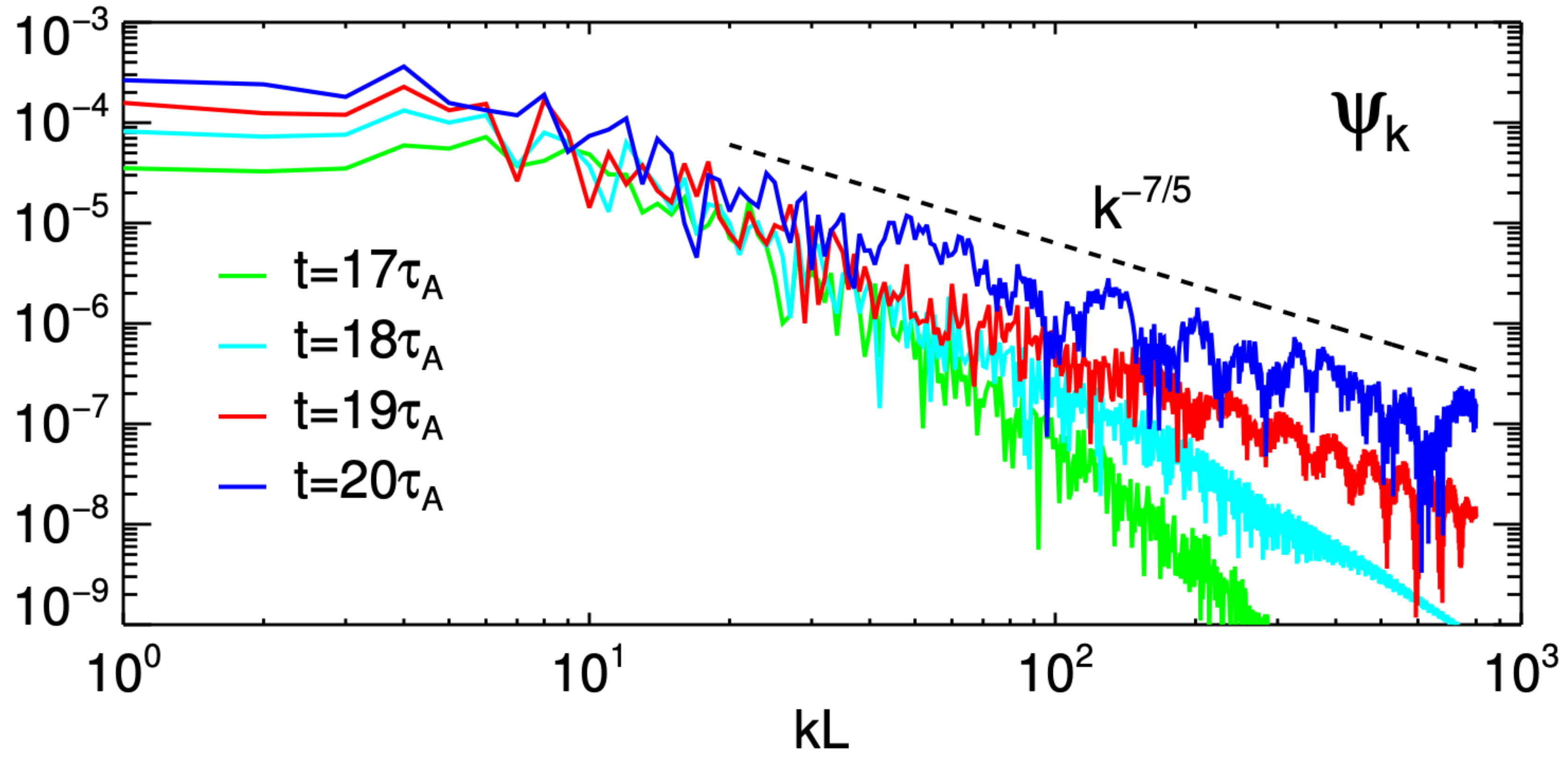
Alfvén time and Lundquist #: $\tau_A^* = L/v_A, \quad (S^*) = L/(v_A \eta)$

Max growth rate: $\gamma \tau_A^* \sim (S^*)^{-1/2}(L/a)^{3/2}$

Self-similar evolution and recurrent collapse







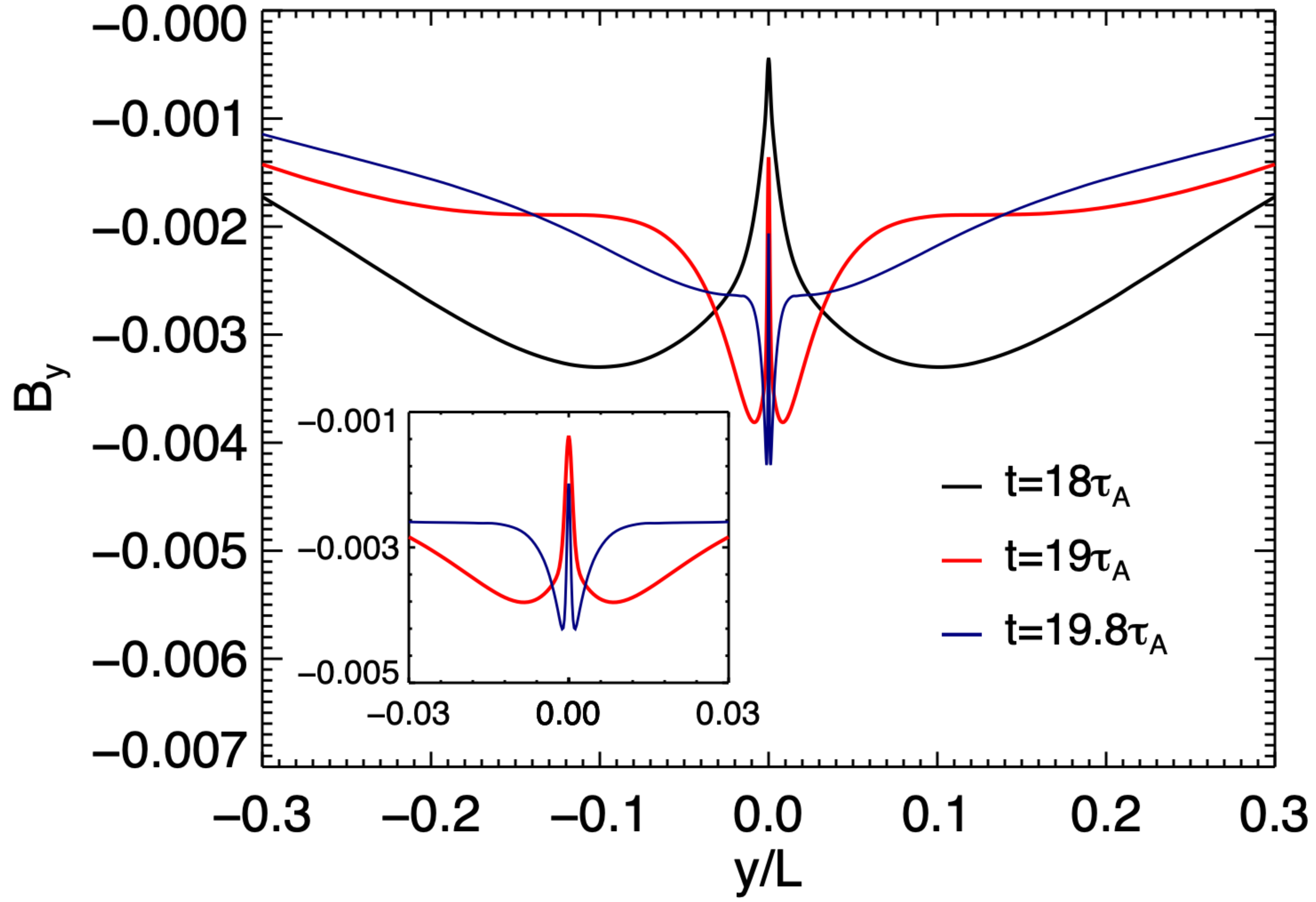
$$a_n \simeq \delta_{n-1}$$

$$a_n/L_{n-1} \sim S_{n-1}^{-1/2}$$

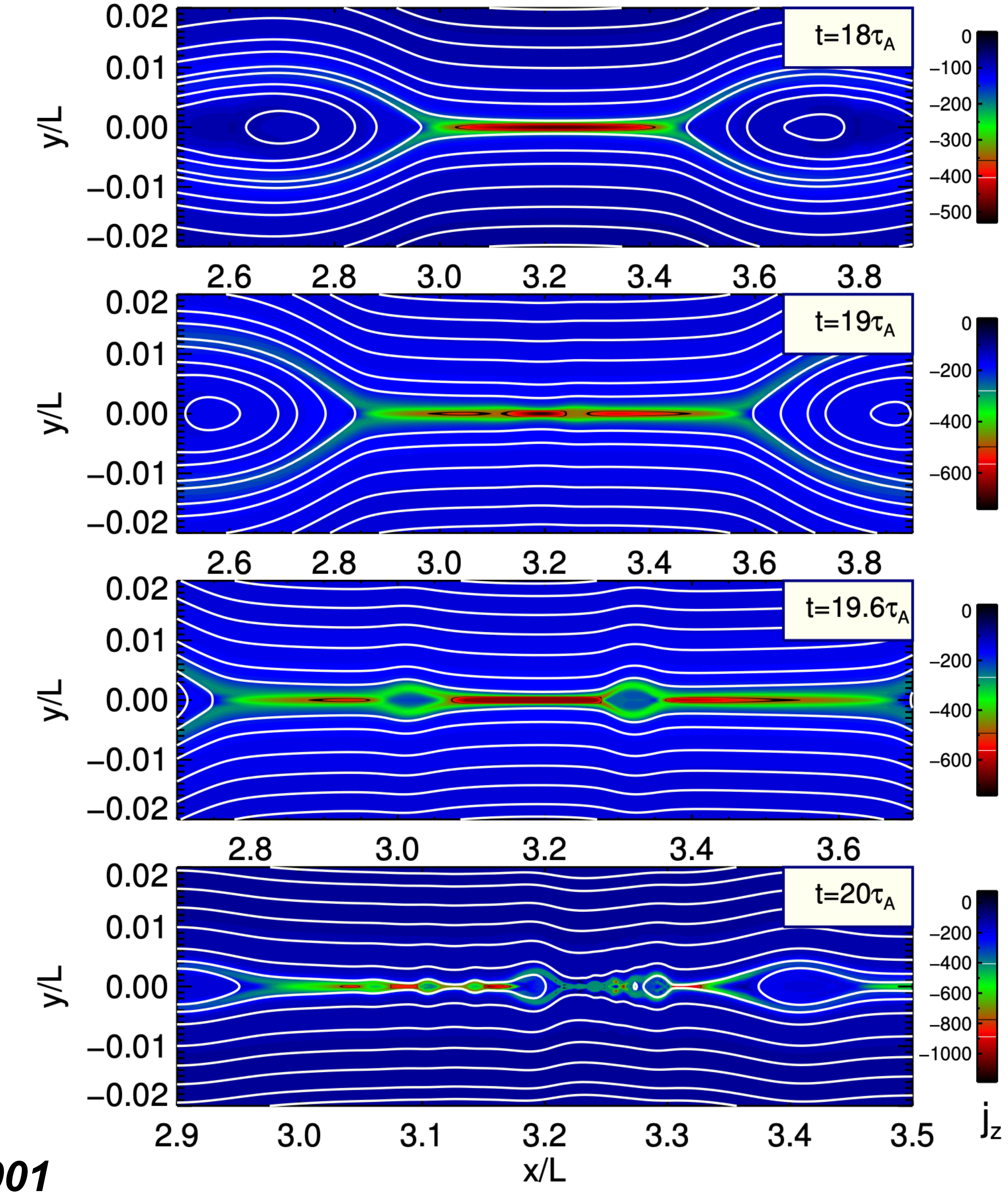
$$a_n/L_n \sim S_n^{-1/3}$$

$$L_n = \frac{L}{S} S_n,\ \tau_{a,n} = \frac{L_n}{L}\tau_a,\ S_n = S^{(3/4)^n}$$

“Ideal tearing” and recursive collapse

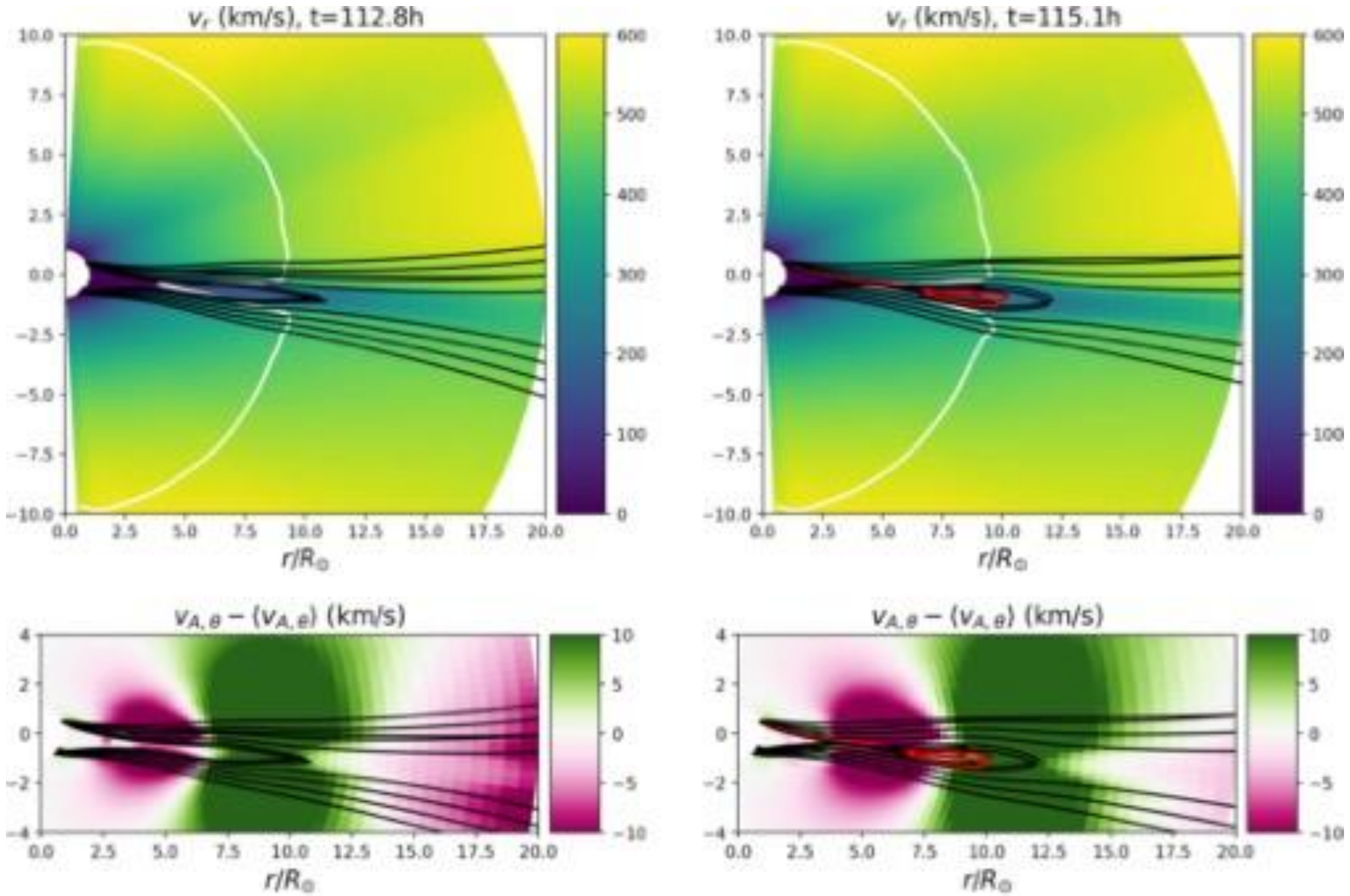


Evidence of “recursive” tearing mode-like instabilities during the nonlinear stage of a primary tearing mode within a Harris current sheet. New plasmoids appear to be generated, at each *n*th step, within smaller and smaller current sheets (CS), that consistently correspond to the inner layer of the (*n*-1)th unstable CS.

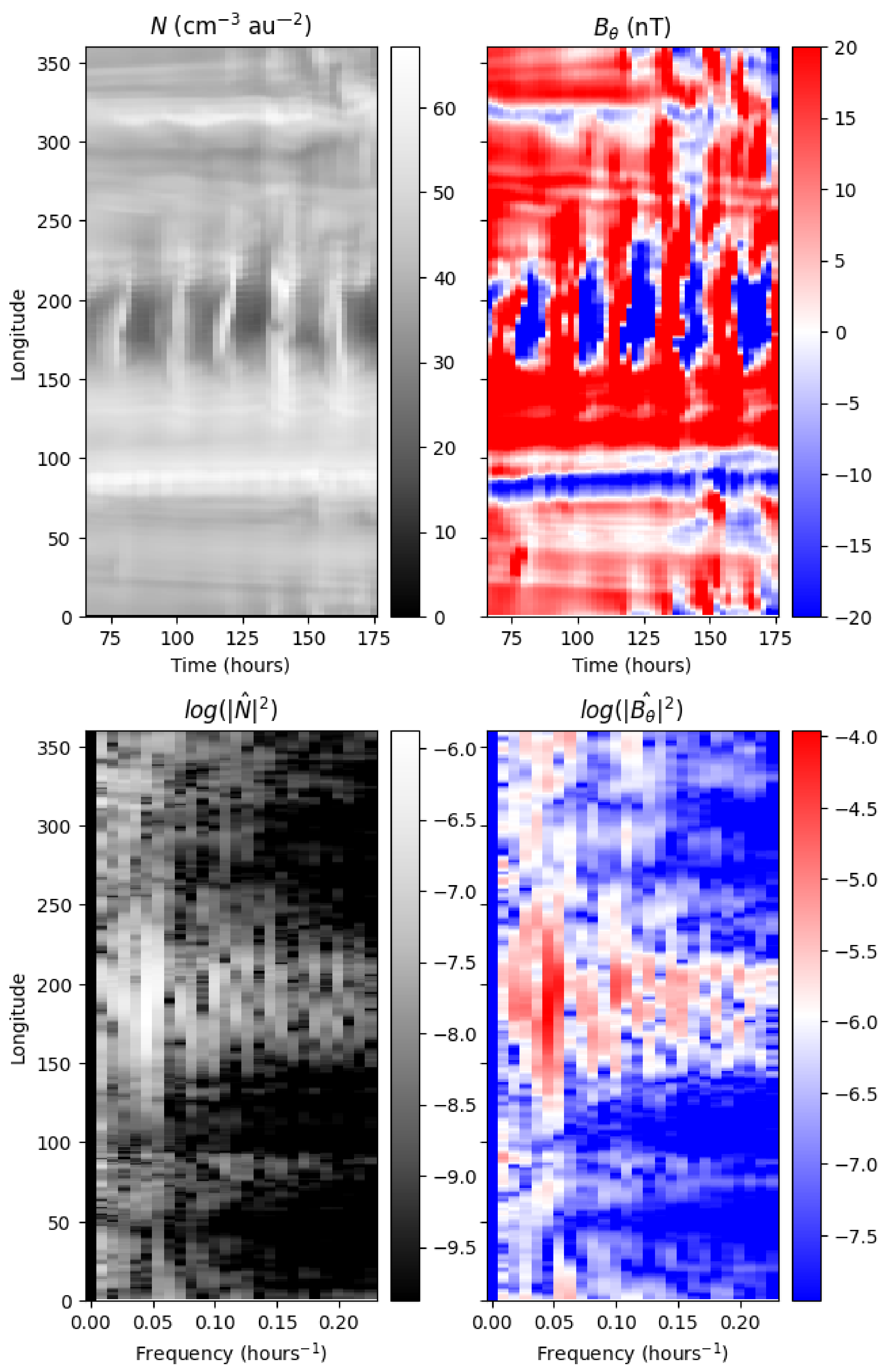


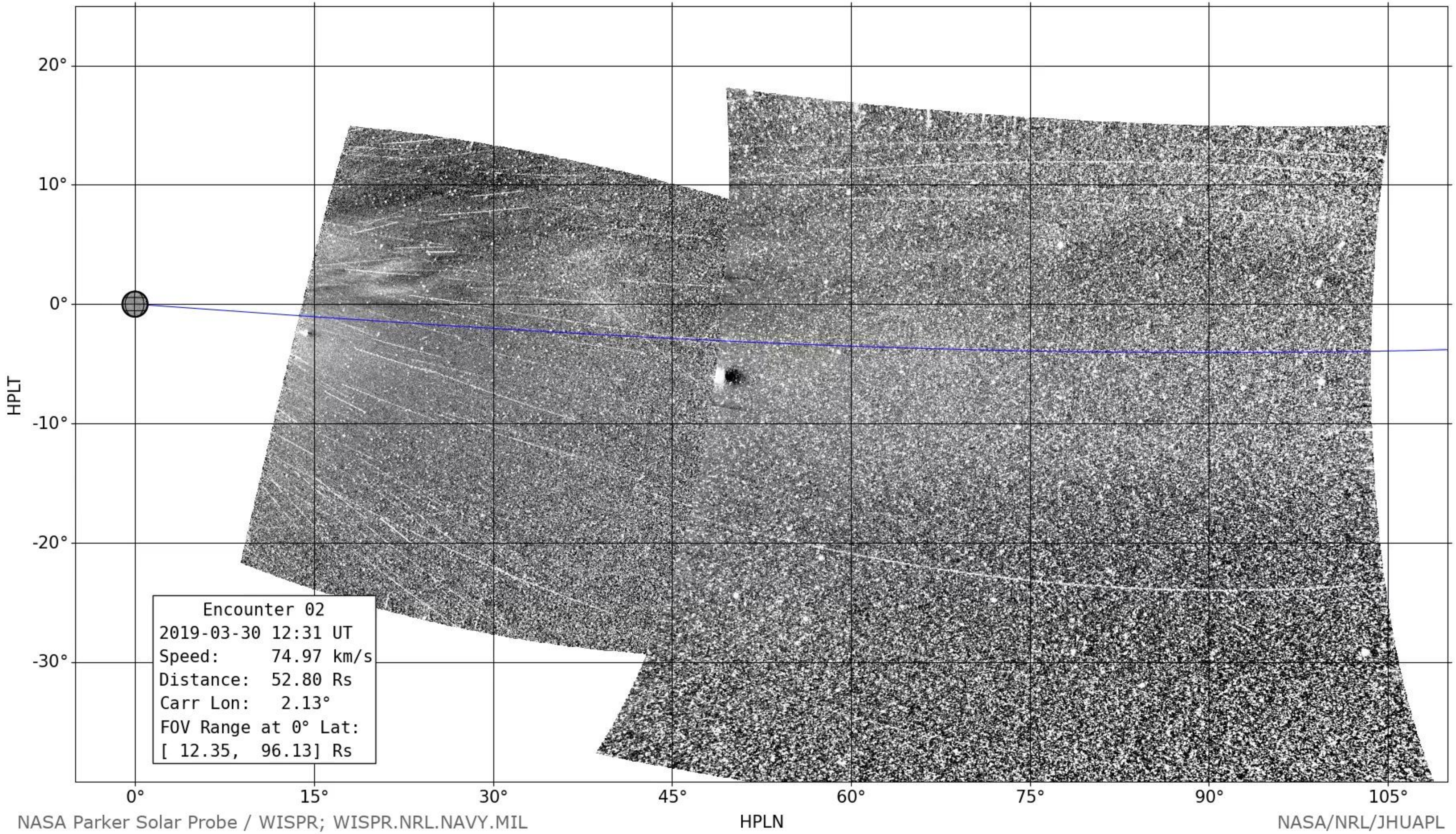
Tenerani et al. 2015b, ApJ inspired by Shibata&Tanuma 2001

Flux ropes in the Heliospheric current sheet



$$\frac{1}{R_c}\frac{\partial^2 \xi}{\partial t^2} = \kappa \cdot (\frac{c_s^2}{p}\nabla p - \mathbf{g})$$




20°
10°
0°
HPLT
-10°
-20°
-30°
Encounter 02
2019-03-30 12:31 UT
Speed: 74.97 km/s
Distance: 52.80 Rs
Carr Lon: 2.13°
FOV Range at 0° Lat:
[12.35, 96.13] Rs
0°
15°
30°
45°
60°
75°
90°
105°
HPLN
NASA Parker Solar Probe / WISPR; WISPR.NRL.NAVY.MIL
NASA/NRL/JHUAPL

In view of the progress in understanding the physical processes in the reconnection region, reduced equations or fully phenomenological models should allow more realistic global simulations in full 3 dimen-sional geometry, which will be the main task in the future.

Here the magnetic activity in the solar corona is a particularly attractive "playground", where one can use, or at least be stimulated by, the stunning obser-vational data from recent satellite missions such as TRACE.

Finally, a word of caution. In spite of the recent advances, the feeling of mastering, after so many years, this scintillating subject called magnetic reconnection might again turn out to be elusive.

Biskamp, 2000

As the images unwind
Like the circles that you find
In the windmills of your mind….
Like a carousel that's turning running rings around the moon
(A. & M. Bergman)