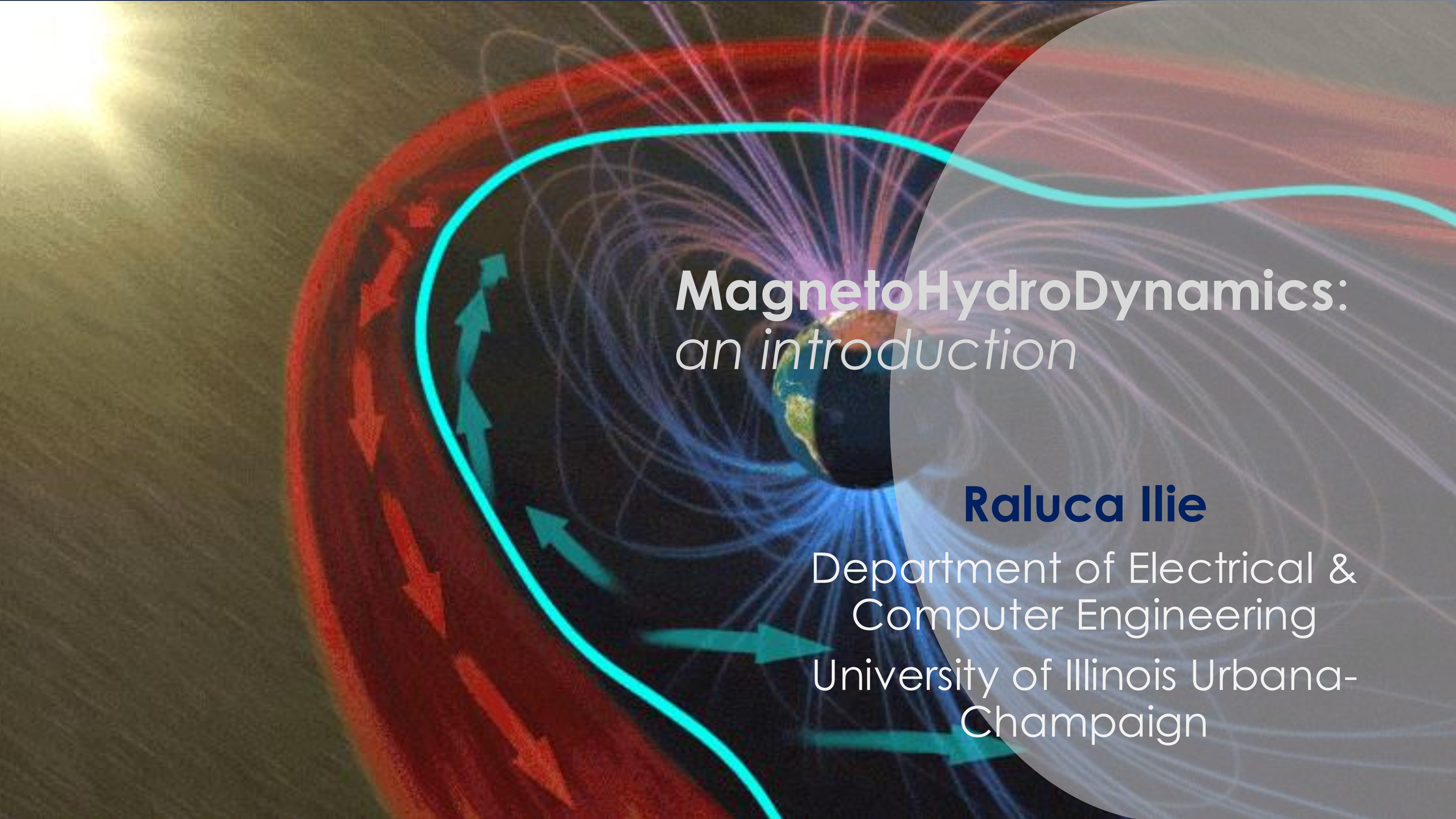

MagnetoHydroDynamics:
an introduction

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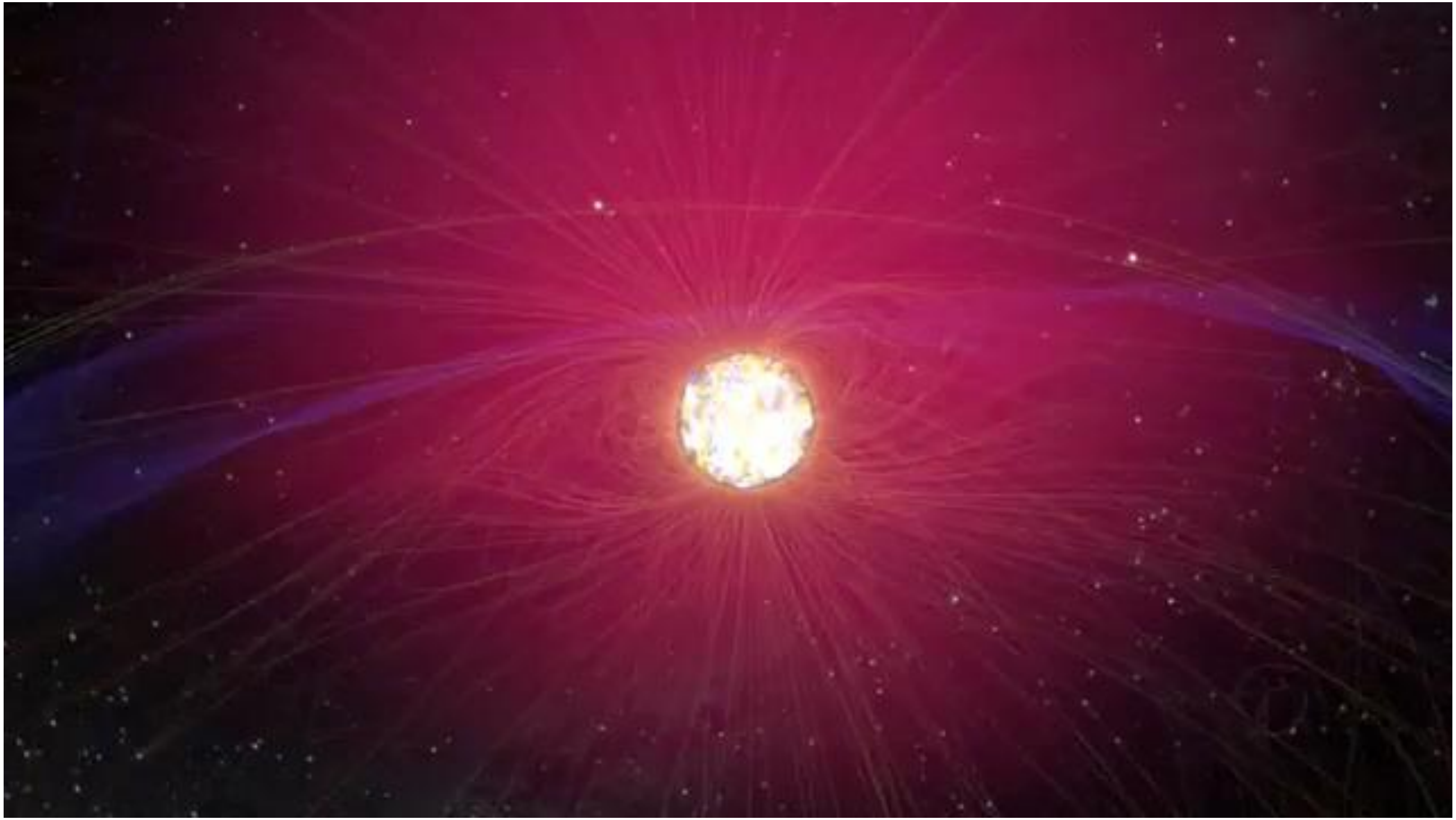
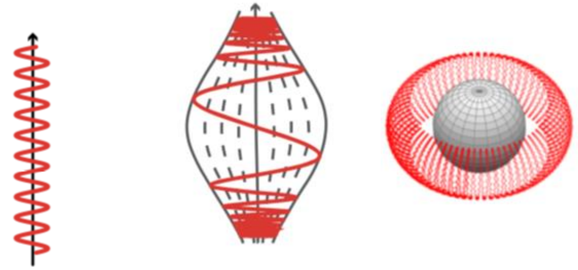
Journey through space



From particles to fluids



- Guiding Center (GC) Theory → single particle motion in complex EM
- If a system contains a large number of particles, GC theory must be augmented with a theory that describes the dynamics of a group of particles.
    - Kinetic theory is capable of deriving transport coefficients from the fundamental properties of the gas molecules. However, this leads to a seven-dimensional partial integro-differential equation→challenging to solve.
- Compressible fluid dynamics describes the fundamental conservation laws for a continuous medium that is composed of individual particles.
    - The particle density, however, is so large that a continuum description is warranted.
    - In fluid dynamics, the classical gas transport coefficients (diffusion coefficient, viscosity, heat conduction) lose their meaning, because the internal state of the gas becomes too complicated, and the coefficients cease to be constant.
- The challenge is how to **connect microscopic and macroscopic** quantities.

From particles to fluids

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Velocity moments

- **Order of a velocity moment**: the sum of the powers of velocity components in the moment integral. For instance:

$$M_6 = \iiint_{\infty} v^2\, v_x^3\, v_y\, F(t, \mathbf{r}, \mathbf{v})\, d^3v$$

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- Zeroth moment: **Density**

$$n(t, \mathbf{r}) = \iiint_\infty F(t, \mathbf{r}, \mathbf{v})\, d^3v$$

- First Moment: **Particle Flux**

$$n(t, \mathbf{r})\, \mathbf{u}(t, \mathbf{r}) = \iiint_\infty \mathbf{v}\, F(t, \mathbf{r}, \mathbf{v})\, d^3v$$

- Second Moment: **Pressure Tensor**

$$P_{ij}(t, \mathbf{r}) = m \iiint_\infty c_i\, c_j\, F(t, \mathbf{r}, \mathbf{c})\, d^3c$$

- Temperature

$$T = \frac{p}{nk} = \frac{m}{k} \frac{1}{n(t, \mathbf{r})} \iiint_\infty c^2\, F(t, \mathbf{r}, \mathbf{c})\, d^3c$$

- Stress Tensor

$$\tau_{ij} = p\delta_{ij} - P_{ij}$$

- Heat Flow

$$\mathbf{h}(t, \mathbf{r}) = \iiint_\infty \left(\frac{1}{2} mc^2\right) \mathbf{c}\, F(t, \mathbf{r}, \mathbf{c})\, d^3c$$

Navier-Stokes equations

- The **zeroth**, **first** and **second moments** of the Boltzmann equation are:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\,\mathbf{u}) = 0$$

$$m\,n\frac{\partial \mathbf{u}}{\partial t} + m\,n\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p - m\,n\,\mathbf{g} - q\,n\,(\mathbf{E} + \mathbf{u}\times\mathbf{B}) = \nabla\cdot\left\{\eta\left[(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T - \tfrac{2}{3}I(\nabla\cdot\mathbf{u})\right]\right\}$$

$$\frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \frac{1}{\gamma-1}(\mathbf{u}\cdot\nabla)\,p + \frac{\gamma}{\gamma-1}p\,(\nabla\cdot\mathbf{u}) = \nabla\cdot(\kappa\nabla T)$$

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- Neglecting the viscous term and heat conduction → **Euler equations**

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Ideal MHD equations: **a combination of Euler and Maxwell's equations**

- Consider a self-consistent description of a conducting fluid and the electromagnetic fields
  - Neglect heat conduction and viscosity
  - Assume quasi-neutrality
  - Neglect the displacement current in Ampére's law

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$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

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- On the other hand

$$\mathbf{j} = \overline{\sigma}_0 \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} \right)$$

$$\nabla \times \mathbf{j} = \overline{\sigma}_0 \left[\nabla \times \mathbf{E} + \nabla \times (\mathbf{u} \times \mathbf{B}) \right]$$

$$\nabla \times (\nabla \times \mathbf{B}) = \overline{\sigma}_0 \mu_0 \left[-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) \right]$$

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- We get the **convection-diffusion** equation

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta_m \nabla^2 \mathbf{B} = -\nabla \cdot (\mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}) + \eta_m \nabla^2 \mathbf{B} \qquad \eta_m = \frac{1}{\overline{\sigma}_0 \mu_0}$$

MagnetoHydroDynamics (MHD) Theory

- Considers the coupled evolution of conducting fluids with the electromagnetic field.
- The evolution is **self-consistent:** the fluid is both influenced by the field, and creates a field due to currents (and charge separation, if present).
- The MHD equations are a combination of the Euler equations and Maxwell's equations.

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \, \mathbf{u}) = 0$$

$$\frac{\partial (\rho_m \, \mathbf{u})}{\partial t} + \nabla \cdot \left(\rho_m \, \mathbf{u}\,\mathbf{u} + p\, I + \frac{B^2}{2\mu_0}\, I - \frac{\mathbf{B}\,\mathbf{B}}{\mu_0} \right) = \rho_m \mathbf{g}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\,\mathbf{B} - \mathbf{B}\,\mathbf{u}) = -\eta_m \nabla^2 \mathbf{B}$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho_m \, u^2 + \frac{3}{2} p + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{1}{2}\, \rho_m \, u^2 \mathbf{u} + \frac{5}{2} p \, \mathbf{u} + \frac{(\mathbf{B}\cdot\mathbf{B})\,\mathbf{u} - \mathbf{B}\,(\mathbf{B}\cdot\mathbf{u})}{\mu_0} \right) = \rho_m \, (\mathbf{g} \cdot \mathbf{u})$$

Continuity equations and conservation laws

$$\frac{\partial}{\partial t}([\chi Density]) + \nabla \cdot ([\chi Flux]) = S$$

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χ is the conserved quantity

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LET'S TALK

Continuity equations and conservation laws

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$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0$$

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$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m u^2 + \frac{3}{2}p + \frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\frac{1}{2}\rho_m u^2 \mathbf{u} + \frac{5}{2}p\,\mathbf{u} + \frac{(\mathbf{B}\cdot\mathbf{B})\,\mathbf{u} - \mathbf{B}\,(\mathbf{B}\cdot\mathbf{u})}{\mu_0}\right) = \rho_m\,(\mathbf{g}\cdot\mathbf{u})$$

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Identify the conserved quantity in each of these equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0$$

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Conservation of mass $\quad \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0$

Conservation of momentum $\quad \frac{\partial (\rho_m \mathbf{u})}{\partial t} + \nabla \cdot \left(\rho_m \mathbf{u}\,\mathbf{u} + p\,I + \frac{B^2}{2\mu_0} I - \frac{\mathbf{B}\,\mathbf{B}}{\mu_0} \right) = \rho_m \mathbf{g}$

Induction equation $\quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\,\mathbf{B} - \mathbf{B}\,\mathbf{u}) = -\eta_m \nabla^2 \mathbf{B}$

Conservation of energy $\quad \frac{\partial}{\partial t}\left(\frac{1}{2}\,\rho_m\,u^2 + \frac{3}{2}p + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{1}{2}\,\rho_m\,u^2\mathbf{u} + \frac{5}{2}p\,\mathbf{u} + \frac{(\mathbf{B}\cdot\mathbf{B})\,\mathbf{u} - \mathbf{B}\,(\mathbf{B}\cdot\mathbf{u})}{\mu_0} \right) = \rho_m\,(\mathbf{g}\cdot\mathbf{u})$

Continuity equations are stronger, locally applied forms of conservation laws!

MHD theory - *recap*

MHD is a **fluid approximation**, and often regarded as the lowest approximation for describing plasmas self-consistently.

- Can be derived either via
  - Fluid Dynamics (*will demonstrate this next*)
  - Kinetic Theory (from either the microscopic plasma equation of the statistic plasma distribution), then take moments of Boltzman equation
- It **only applies for large length and time scales** that allow us to ignore single-particle motion and displacement current.
  - Plasmas in space are much more rarified compared to "regular" fluids.
  - The fluid behavior stems not from "billiard ball collisions" but from the collective interaction at a distance due to electromagnetic forces between the particles.
  - Free charges do not accumulate, since the systems is assumed to be a good conductor

Limitations of MHD equations

Inherent difficulty: more variables than equations

- the n*th* velocity moment also depends on the components of the (n+1)*th* moment

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For instance:

0^{th} moment eq.

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0$$

1^{st} moment eq.

$$\frac{\partial (\rho_m \mathbf{u})}{\partial t} + \nabla \cdot \left(\rho_m \mathbf{u}\,\mathbf{u} + p\, I + \frac{B^2}{2\mu_0}\, I - \frac{\mathbf{B}\,\mathbf{B}}{\mu_0} \right) = \rho_m \mathbf{g}$$

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  - **there is no electric field in the rest frame**
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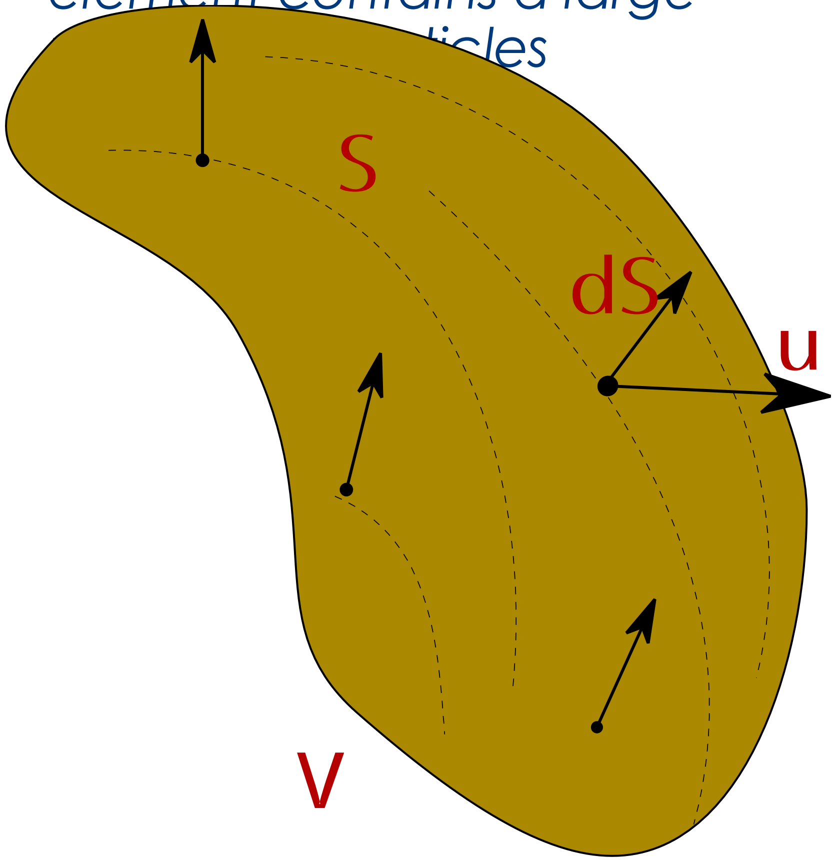
Non-ideal MHD descriptions exist which include finite resistivity, heat conduction, and charge separations (modified MHD equations, still fluid description!!)

Concepts in Fluid Dynamics: **conservation of mass**

MHD fluid treated as a continuum

- *any small volume in a fluid element contains a large*
  ticles



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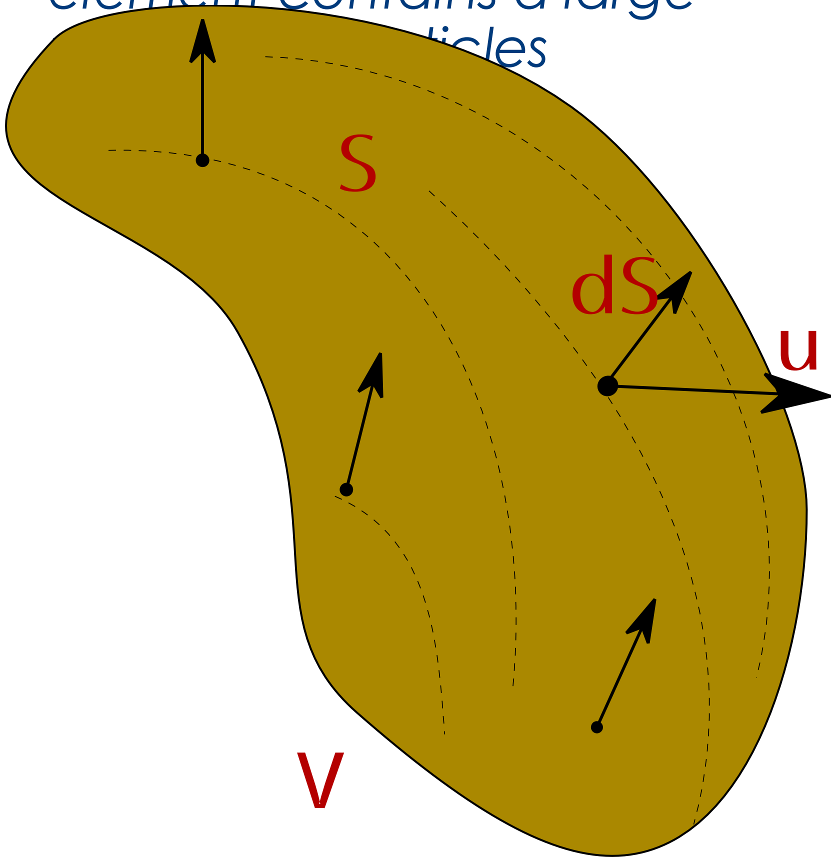

- Given a macroscopic fluid with density ρ, velocity **u**(*x*, *y*, *z*, *t*) of the fluid element at (x, *y*, z) and time *t*.
- For a fluid displaced in a time *dt* a distance **u** *dt* → the mass of the fluid crossing the surface element *dS* per unit time is $\rho u \cdot \hat{n}\, dS\, t$
- **Total mass**, *assuming a closed system with no sources nor sinks,*

$$m = \int_S \rho \mathbf{u} \cdot \hat{n} dS = -\int_V \frac{\partial \rho}{\partial t} dV = \int_V \nabla \cdot (\rho \vec{u}) dV$$

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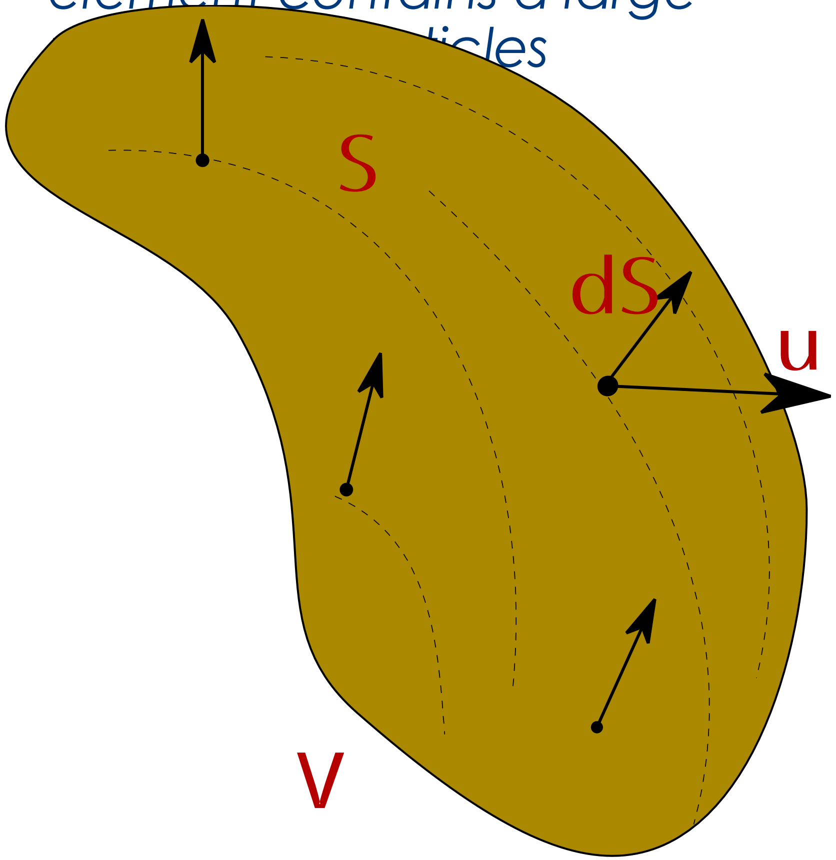

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Mass continuity equation:

$$\nabla \cdot (\rho \mathbf{u}) + \frac{\partial \rho}{\partial t} = 0$$

- *valid for all fluids, independent of their nature (adiabatic, isothermal, compressional, turbulent, etc.)*

Continuity equation

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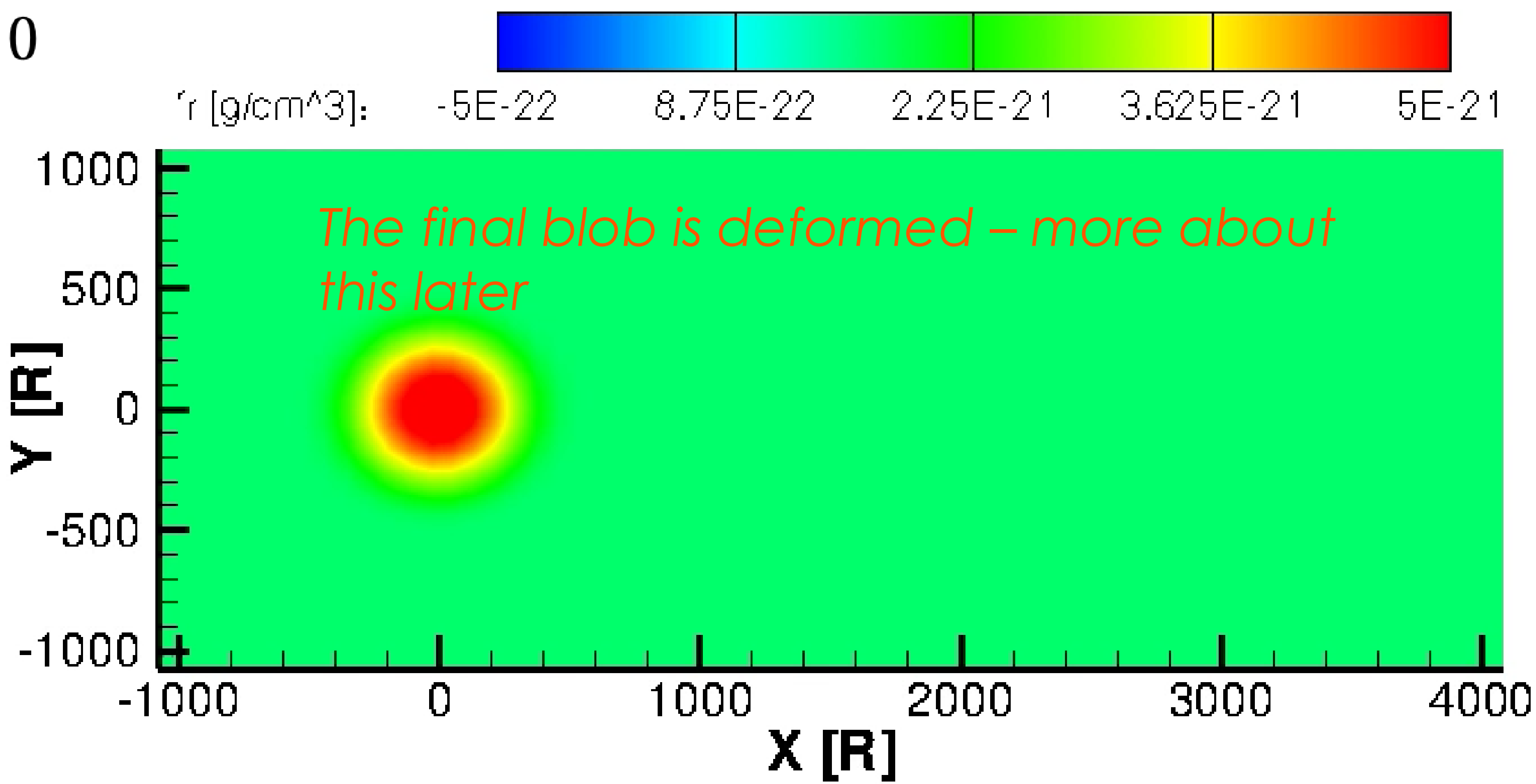

***Initial condition**: a blob of dense plasma moving in the x-direction (to the right) with uniform speed.*

Momentum equation

- Equation of motion for a single particle moving with velocity **v** in an EM field

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- Assuming no thermal motions and no collisions→ all n particles move together with fluid **u: *equation for the force density***

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- But thermal motion gives rise to pressure *p*, and assuming the charged particle fluid acts as an ideal gas ($p = nkT$ and the pressure is not uniform)

$$nm\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p$$

  - The pressure is isotropic → *p* is scalar.

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- The equation of motion for each species:

$$m_i n_i \frac{\mathrm{d}\mathbf{u_i}}{\mathrm{d}t} = q_i n_i(\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) - \nabla p$$

$$m_e n_e \frac{\mathrm{d}\mathbf{u_e}}{\mathrm{d}t} = q_e n_e(\mathbf{E} + \mathbf{u_e} \times \mathbf{B}) - \nabla p$$

charge neutrality: $n = n_e + n_i,\ q_i = -q_e = |q|$

total pressure: $p = p_e + p_i$

current density: $\mathbf{J} = n_i q_i \mathbf{u_i} + n_e q_e \mathbf{u_e}$

- **Momentum equation:**

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{J} \times \mathbf{B} - \nabla p$$

Momentum equation

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{J} \times \mathbf{B} - \nabla p$$

Momentum equation

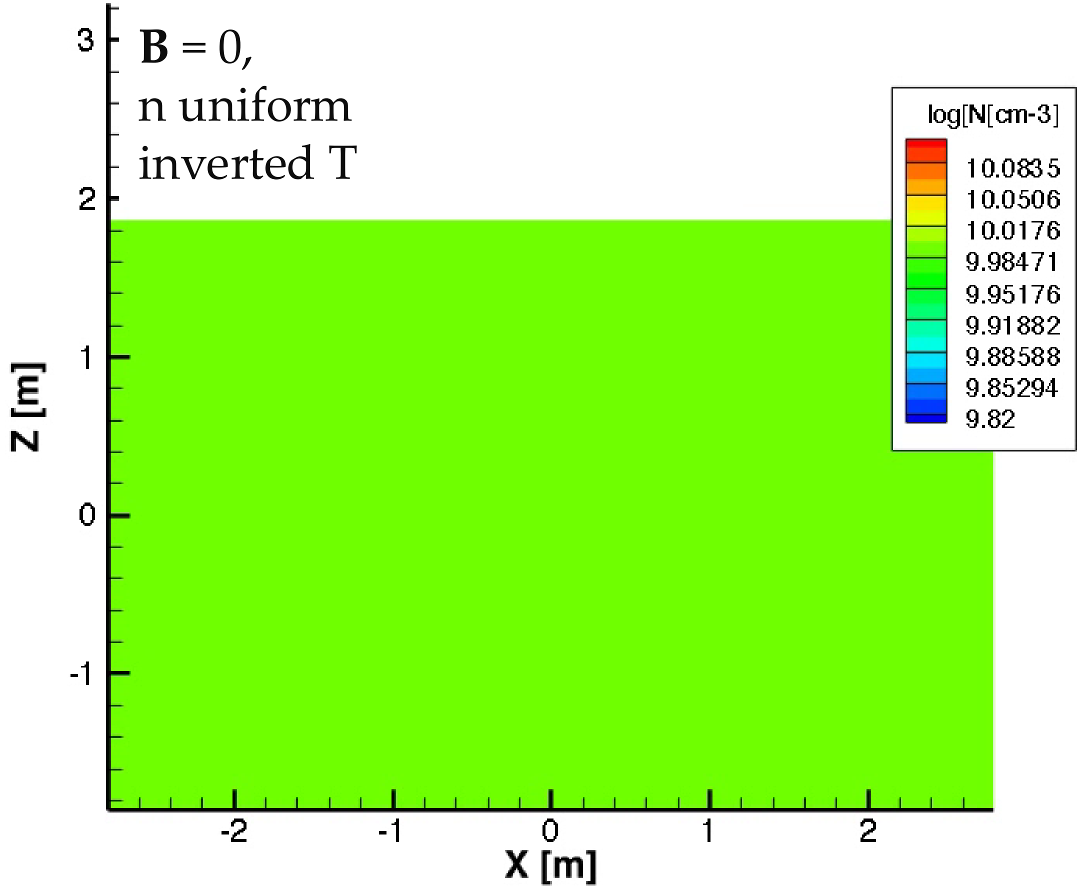
$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{J} \times \cancelto{0}{\mathbf{B}} - \nabla p$$

no ambient magnetic field

Momentum equation

$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{J} \times \overset{0}{\cancel{\mathbf{B}}} - \nabla p$$

no ambient magnetic field



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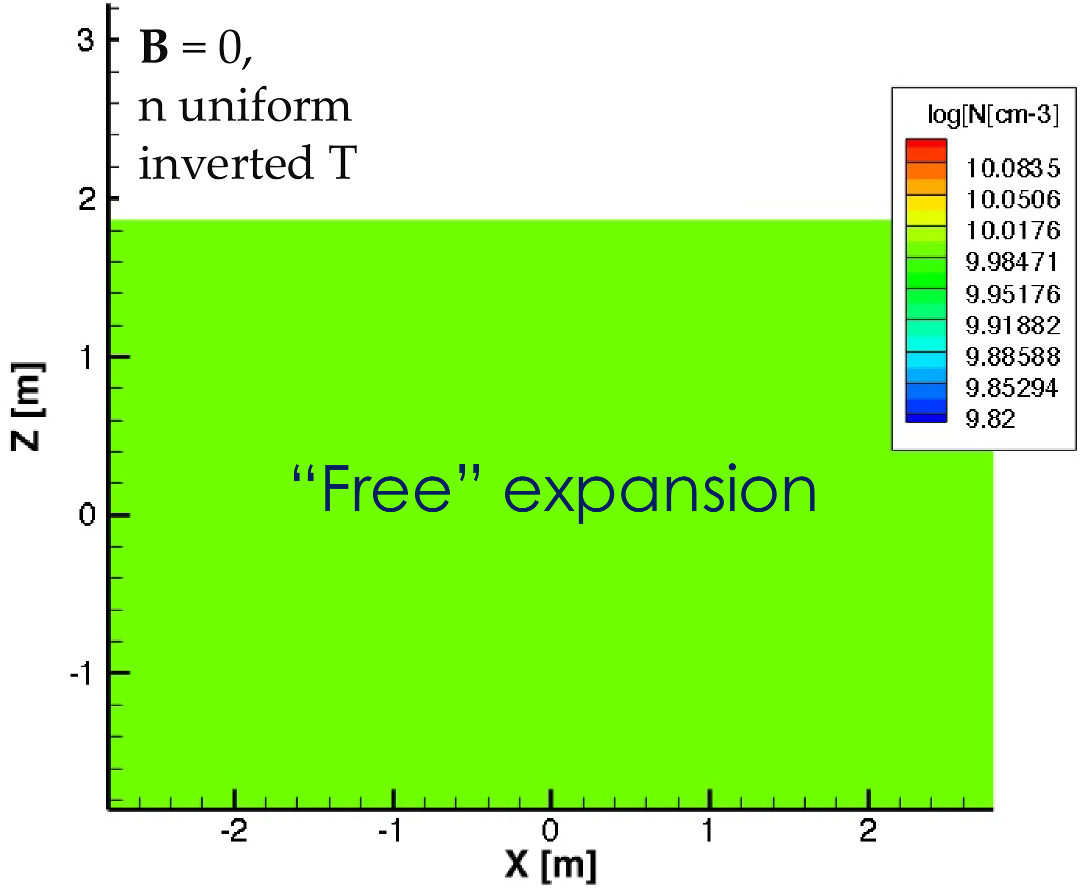
no ambient magnetic field



Momentum equation

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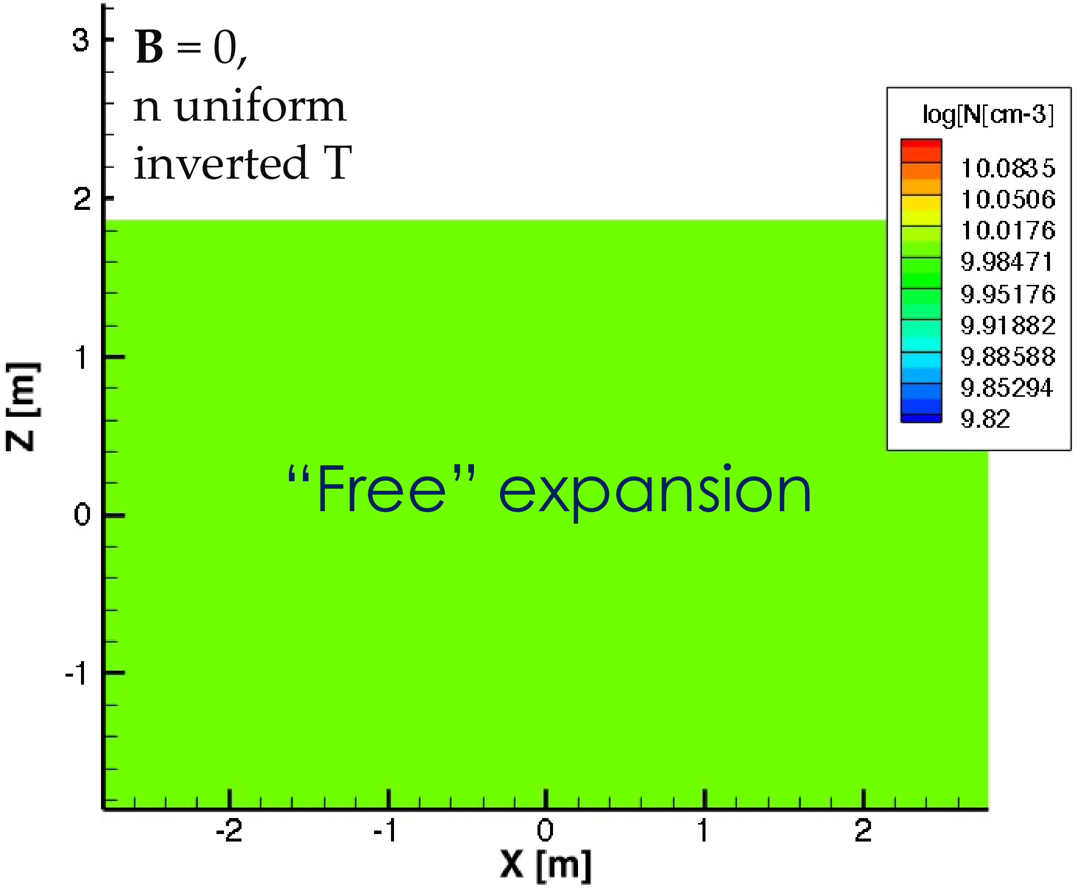

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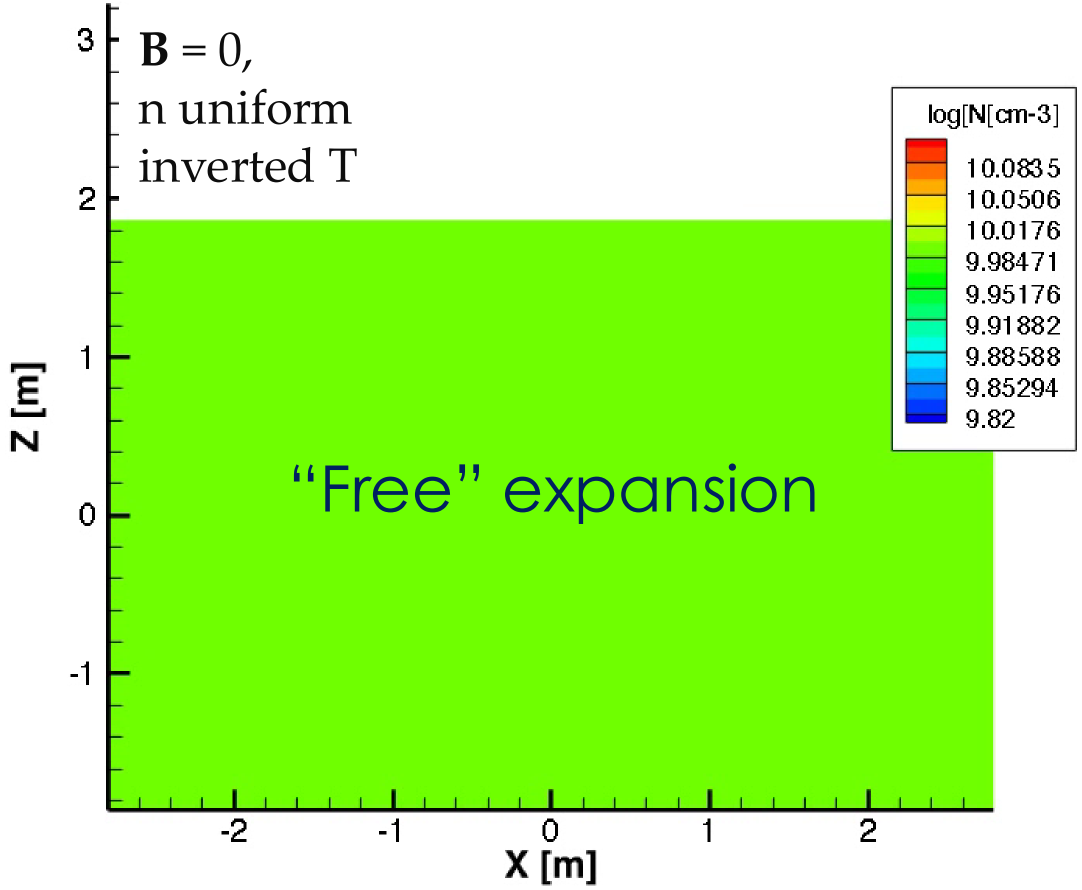

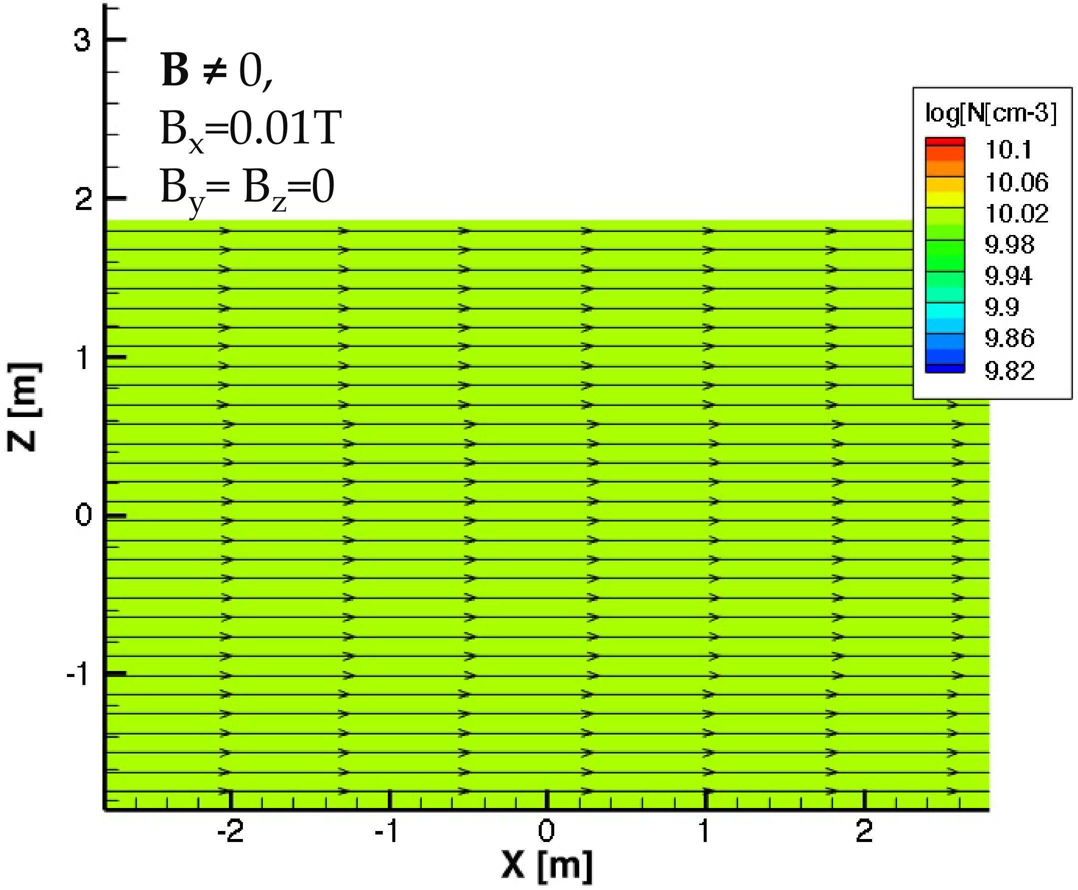

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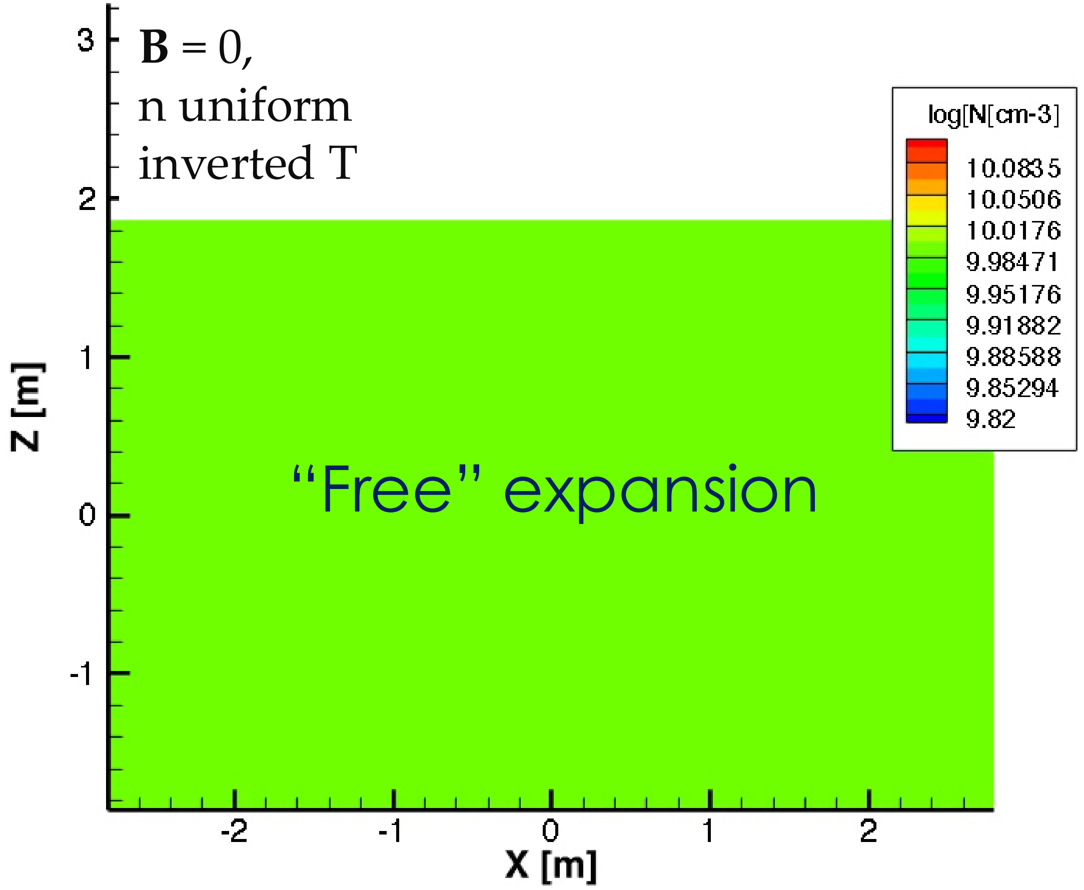
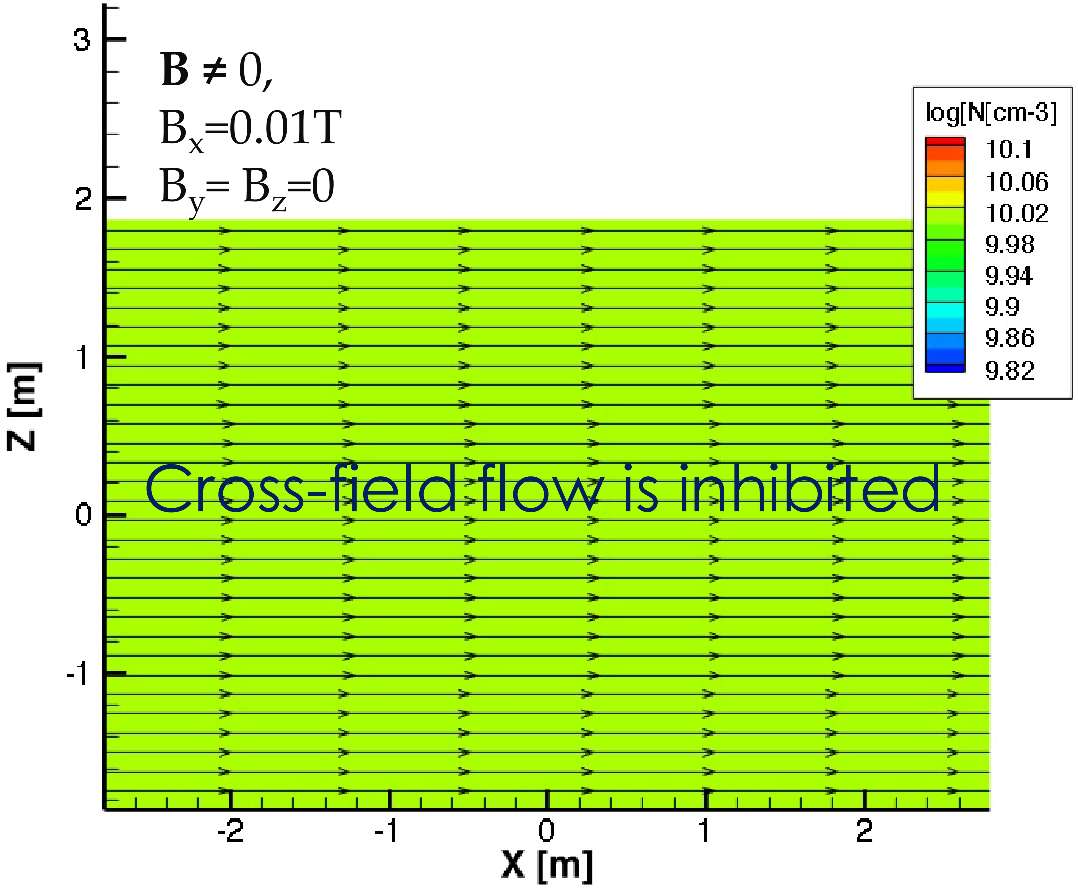


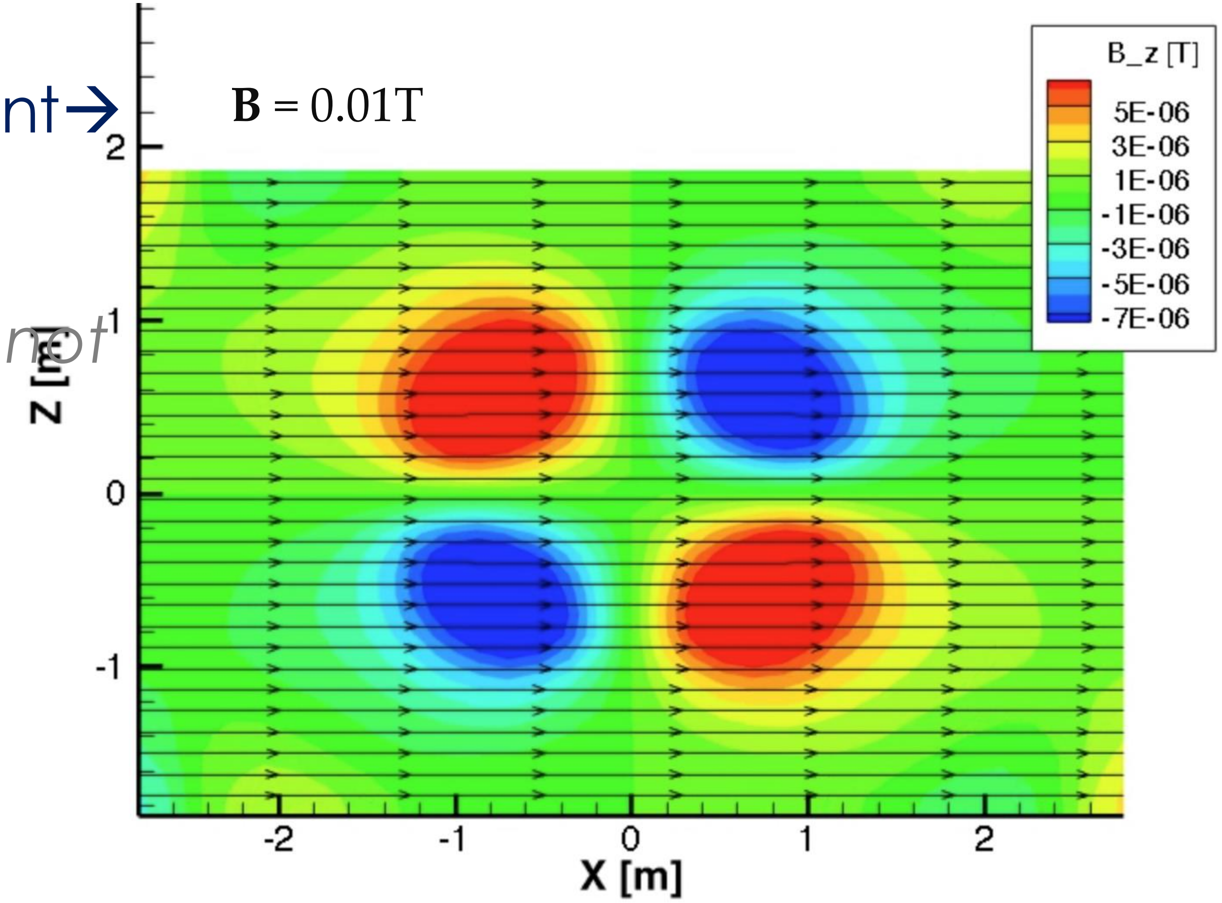


Momentum equation

We can understand $\rho \frac{\mathbf{du}}{\mathbf{d}t} = \mathbf{J} \times \mathbf{B} - \nabla p$ in terms of currents:

- The expansion ***distorts*** the frozen-in magnetic field, creating a z-component→ $B_z \neq 0$

Note that the "hill" shaped field lines are not visible seen since $B_z << B_x$!!



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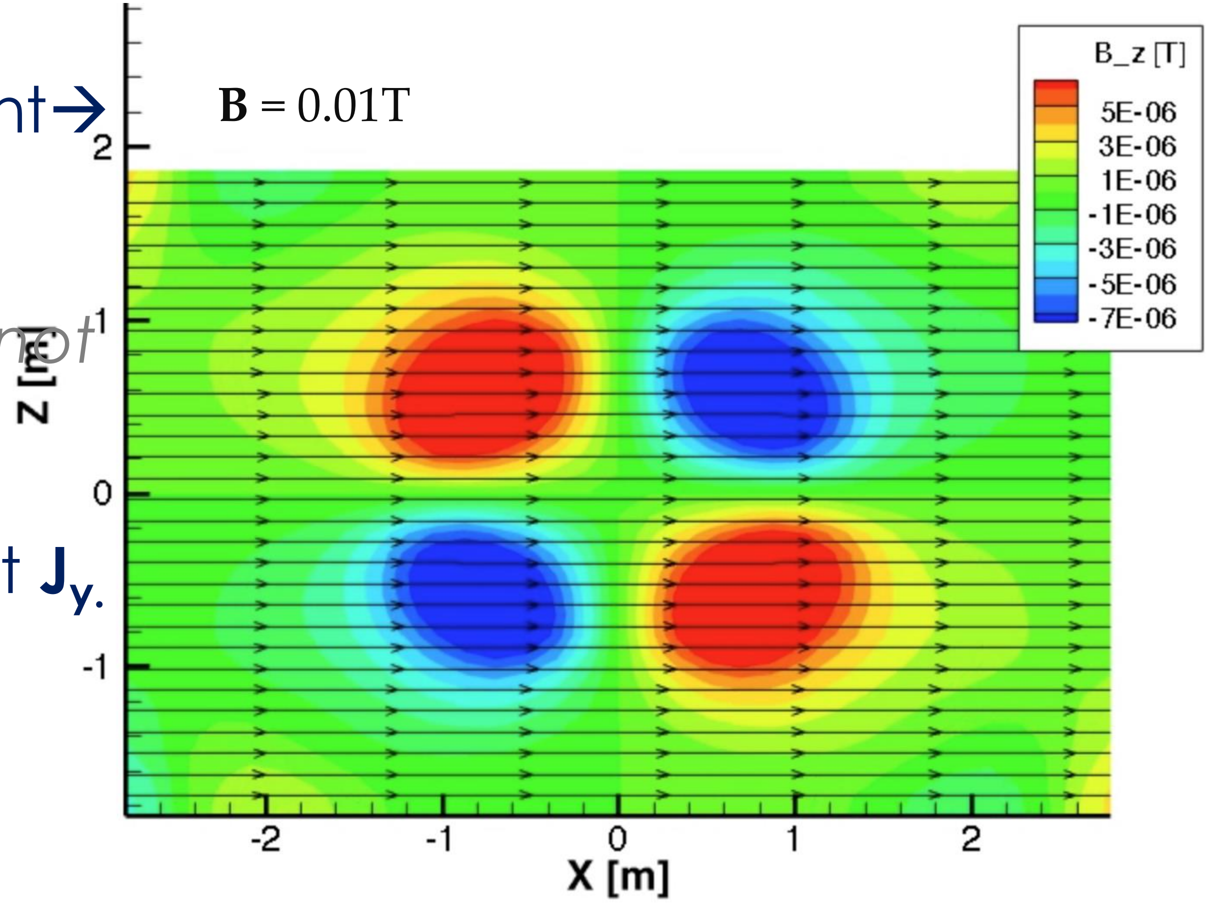
- A **J x B** (Lorentz) force arises in the z-*direction*, opposing the expansion.



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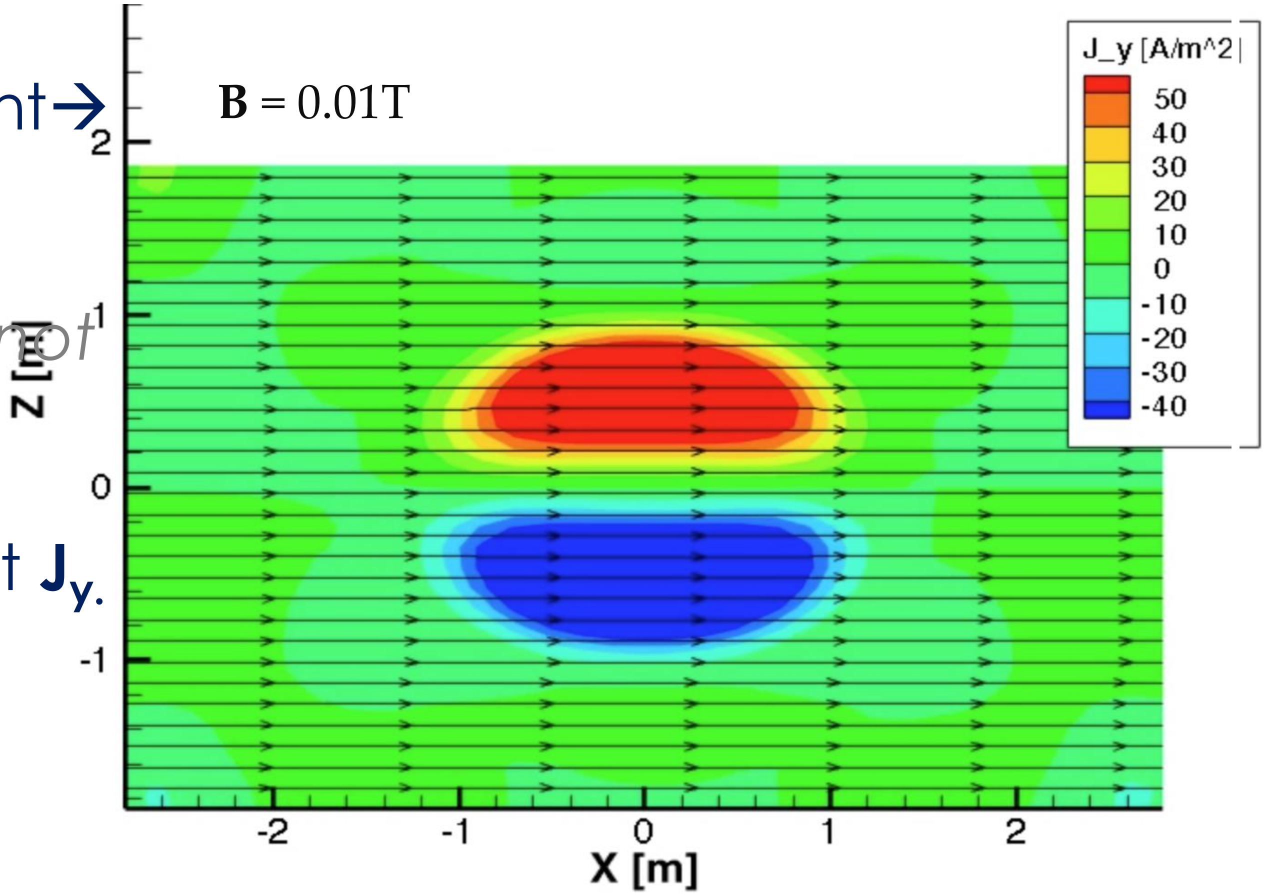

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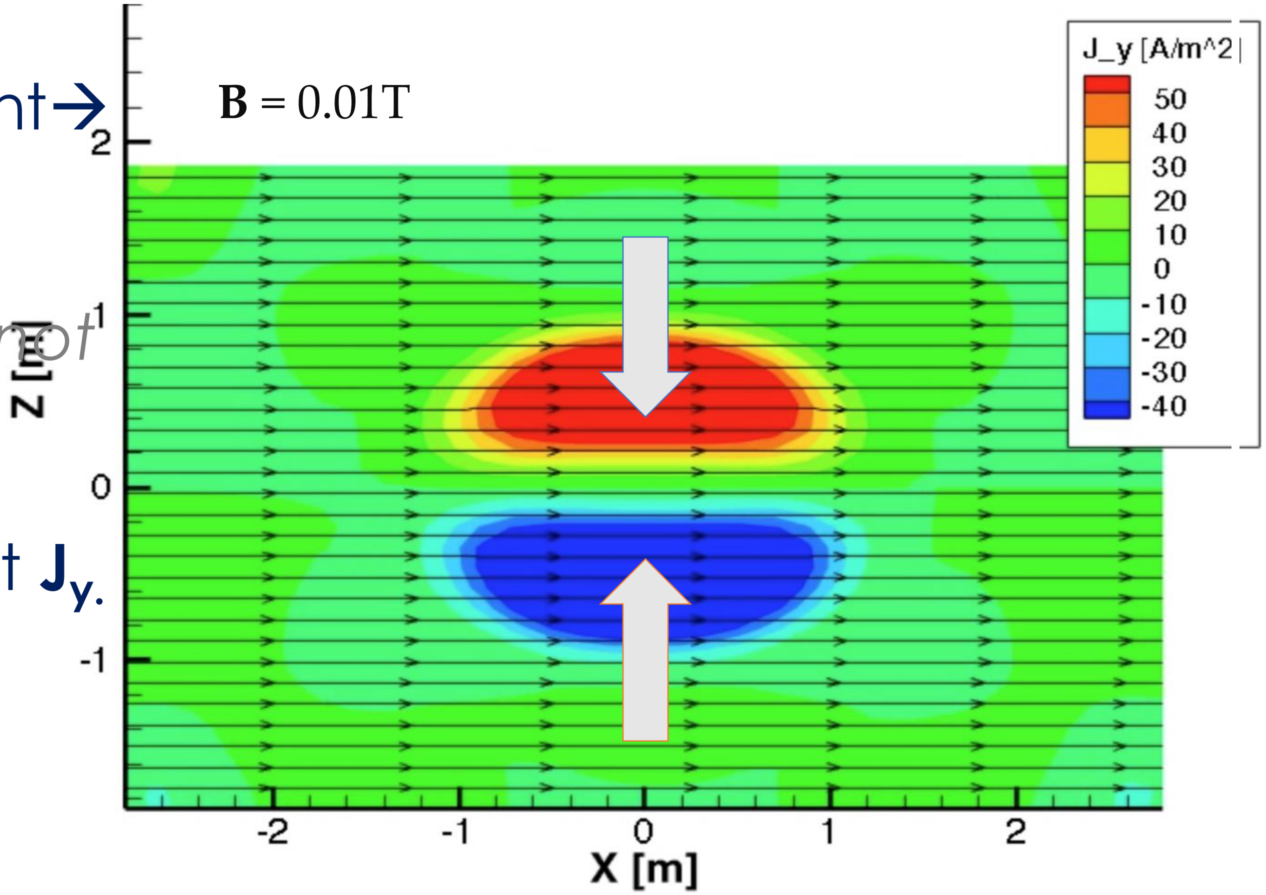

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[*Re-arranged*]→includes the magnetic pressure and tension that act on the fluid

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot \left(\rho\mathbf{uu} + pI - \frac{B^2}{2\,\mu_0} I - \frac{\boldsymbol{BB}}{\mu 0}\right) = \rho\mathbf{g}$$

Continuity and Momentum equations

- They apply all fluids, regardless of their nature, and are fundamental laws of physics.
- Contain 15 independent vars ($\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$, $\mathbf{u}$, ρ, ρ_e, p) but only 11 independent eqns.
- Additional equations are obtained by making assumptions on the nature of the fluid.
- Assume that the fluid acts like a conductor → use Ohm's law (adds 3 more eqns.)
- Assign a thermodynamic equation of state to the fluid (ads 1 more equation).

The closed set of MHD equations REQUIRES restrictive assumptions!

Low-frequency Maxwell's equations

Neglecting displacement current in Maxwell's equations implies that MHD deals with low frequency phenomena.

$$\nabla \times \mathbf{H} = \mathbf{J} + \cancelto{0}{\frac{\partial \mathbf{D}}{\partial t}}$$

- Note that $|\frac{\partial \mathbf{D}}{\partial t}| = \epsilon_0 |\frac{\partial \mathbf{E}}{\partial t}| \approx \epsilon_0 \frac{|\mathbf{E}|}{T}|$,where T is the characteristic time variation of EM quantities
- MHD requires $\epsilon_0 \frac{|\mathbf{E}|}{T} \ll \mathbf{J} = |\nabla \times \mathbf{H}| \approx \frac{H}{L} = \frac{B}{\mu_0 L}$, where L is the characteristic spatial variation of EM quantities
- Highly conducting fluid implies $\frac{\mathbf{J}}{\sigma} \to 0, \text{ then } |\mathbf{E}| = |\mathbf{u} \times \mathbf{B}| \text{ and } \epsilon_0 \frac{|\mathbf{E}|}{T} \ll \mathbf{J}$
- This means that $\epsilon_0 \frac{uB}{T} \ll \frac{B}{\mu_0 L}$ and $\boxed{T \gg \frac{uL}{c^2}}$
- For most cases, $\frac{u}{c} \ll 1$, therefore displacement current can be ignored if $\boxed{T \gg \frac{L}{c}}$

Why does MHD ignores charge density?

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- Let's consider the conservation of charge:

$$\frac{\partial \rho_e^+}{\partial t} + \nabla \cdot \rho_e^+ \vec{u}^+ = 0$$

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- Adding them we get

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

where the total current density is given by:

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$$\mathbf{J} = \sigma \mathbf{E}$$

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$$\frac{\partial \rho_e}{\partial t} + \sigma \nabla \cdot \mathbf{E} = \frac{\partial \rho_e}{\partial t} + \frac{\sigma}{\epsilon_0} \rho_e = 0$$

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LET'S TALK

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$$\rho_e = \rho_0 e^{(-\frac{\sigma}{\epsilon_0})t}$$

Charge density rapidly decays with time!

Energy equation

- Starting from the momentum and continuity equations, assuming the MHD fluid to be adiabatic and σ → ∞, we obtain (*with effort*) the **energy equation:**

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2\mu_0}\right)+\nabla\cdot\left(\frac{\rho u^2}{2}\mathbf{u}+\frac{\gamma}{\gamma-1}p\mathbf{u}+\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)=0$$

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Note: The energy equation gives additional information and in many MHD cases is **not needed for closure**.

Ideal MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\mathbf{u} + pI + \frac{B^2}{2\mu_0} I - \frac{\mathbf{B}\mathbf{B}}{\mu_0} \right) = \rho \mathbf{g}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{3}{2} p + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{1}{2}\rho u^2 \mathbf{u} + \frac{5}{2} p\mathbf{u} + \frac{(\mathbf{B}\cdot\mathbf{B})\mathbf{u} - \mathbf{B}(\mathbf{B}\cdot\mathbf{u})}{\mu_0} \right) = \rho(\mathbf{g} \cdot u)$$

- Goal #1: Name each of the equation above.
  - Each of them is a conservation law of a different physical quantity. The conserved quantity appears in the time-derivative in each of the equations.
- Goal #2: Describe each of the terms in the equations.
  - For each of these four conservation law, you should be able explain how each term works to change the conserved quantity.
  - For each equation, compare the rank of all terms (i.e. are they scalar, vector or 2D-tensor).

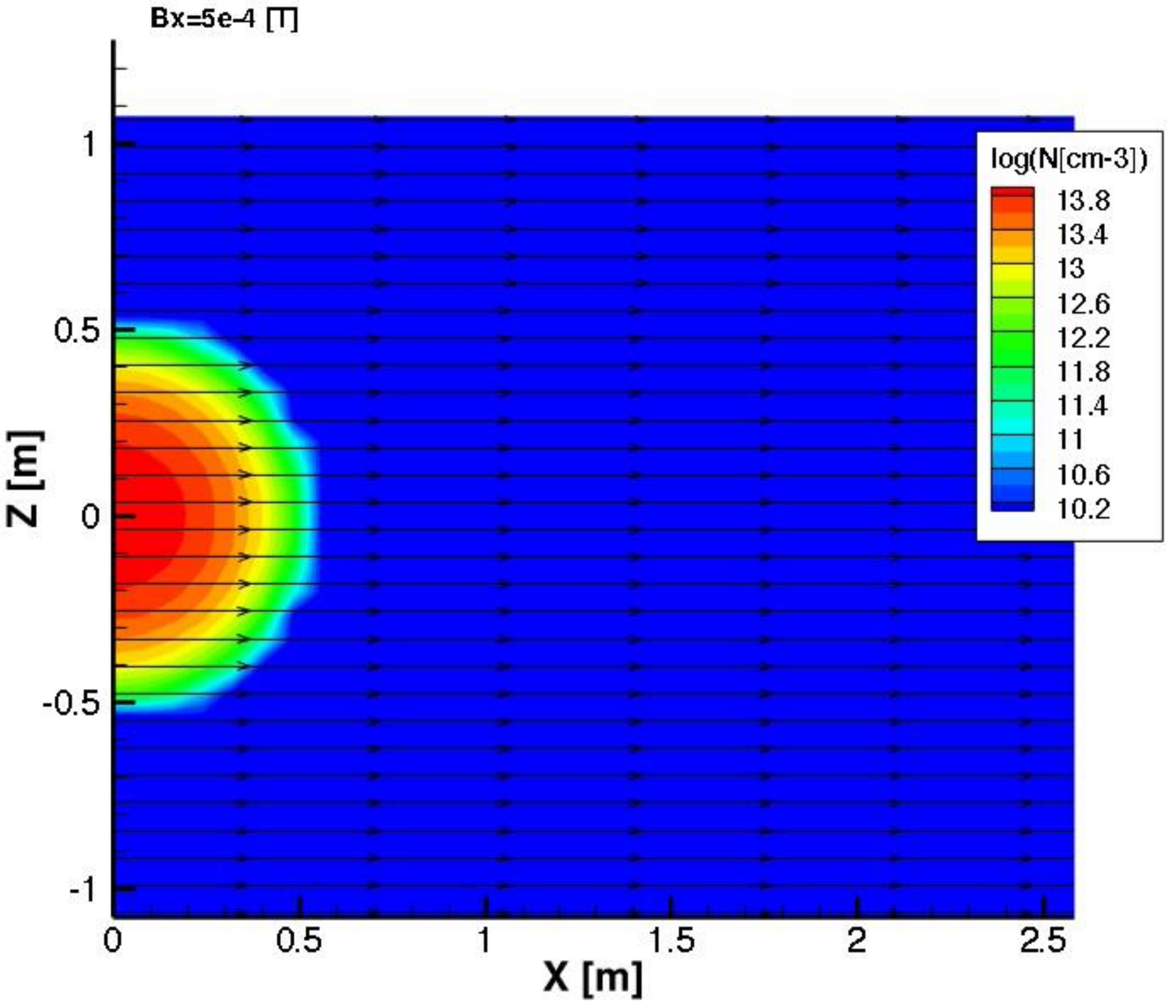
Induction equation

- Describes the evolution of the magnetic field due to fluid motions at bulk velocity **u**:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{uB} - \mathbf{Bu}) = 0$$

Exercise: Assume a plasma of uniform density and pressure, moving in the x-direction at uniform speed $\mathbf{u} \cdot \hat{\mathbf{x}}$. The initial ambient magnetic field **B** =(0,0,B_0(x)) in a bounded region - 1 < x < 1 and **B** = 0 everywhere else. Use the induction equation to calculate the magnetic field as a function of position (x) after T seconds.



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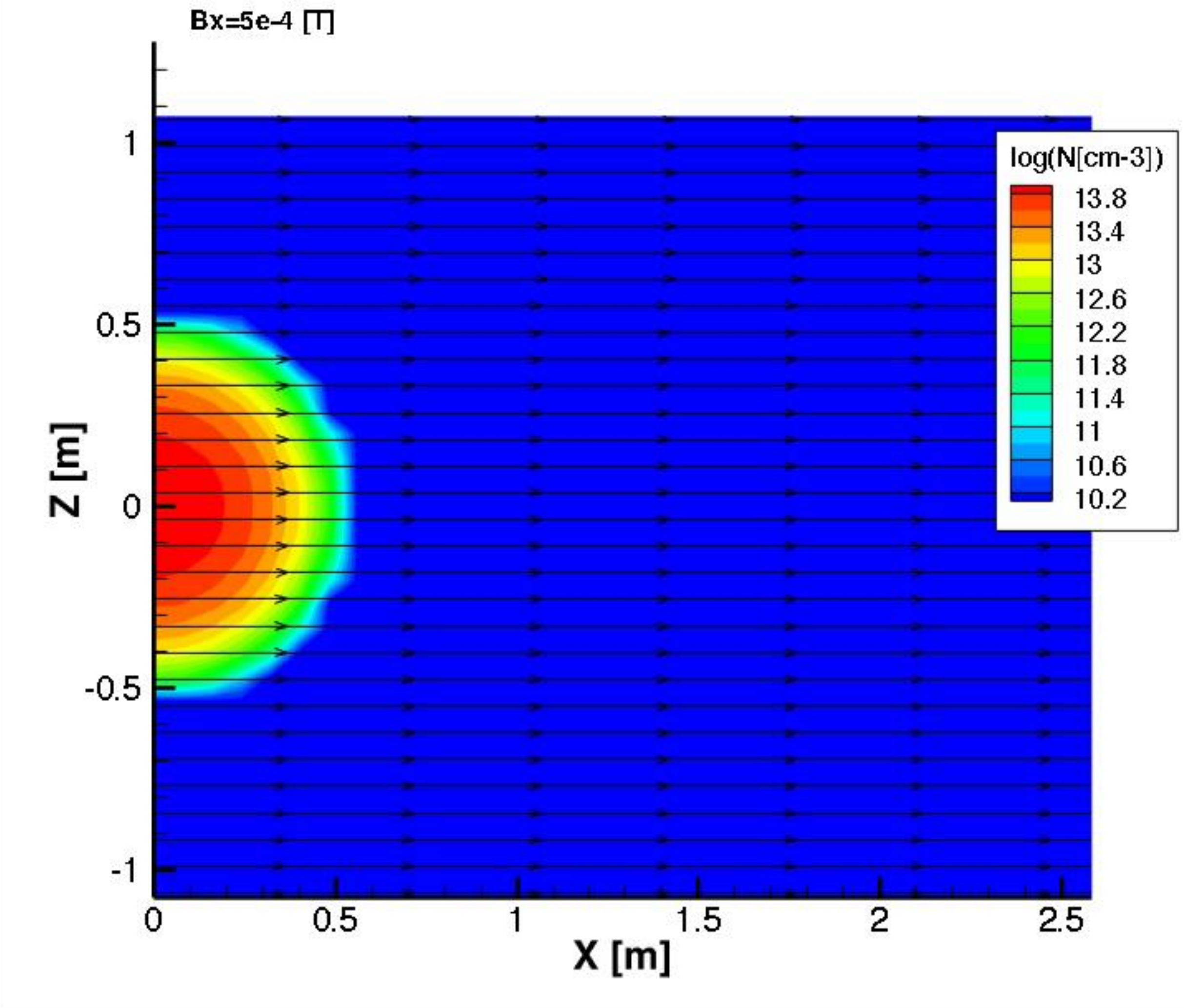

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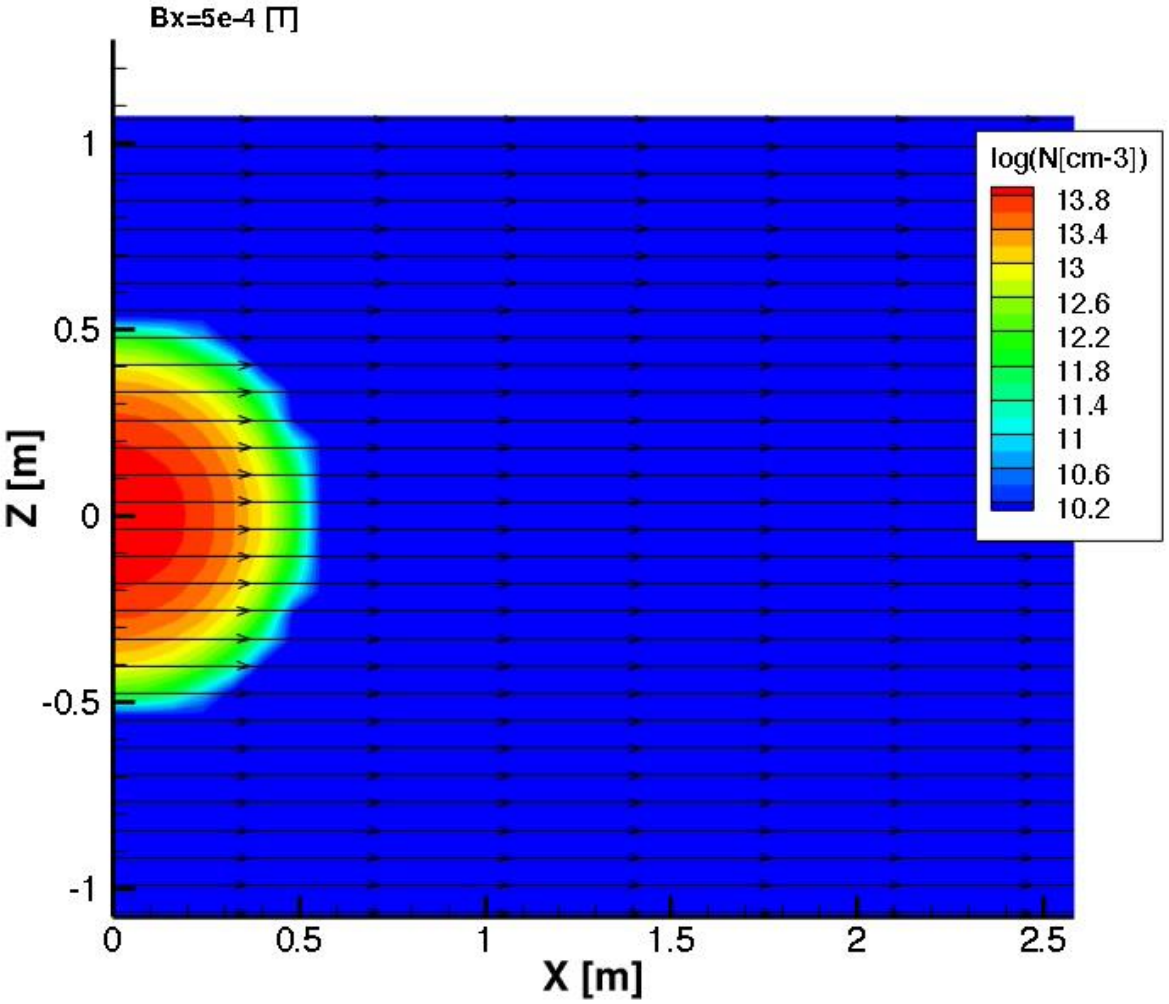
- Infinite conductivity → the **field is frozen** into plasma.




LET'S TALK

Imagine a world where electrical charges have only one polarity. Describe it!

Ideal MHD equations

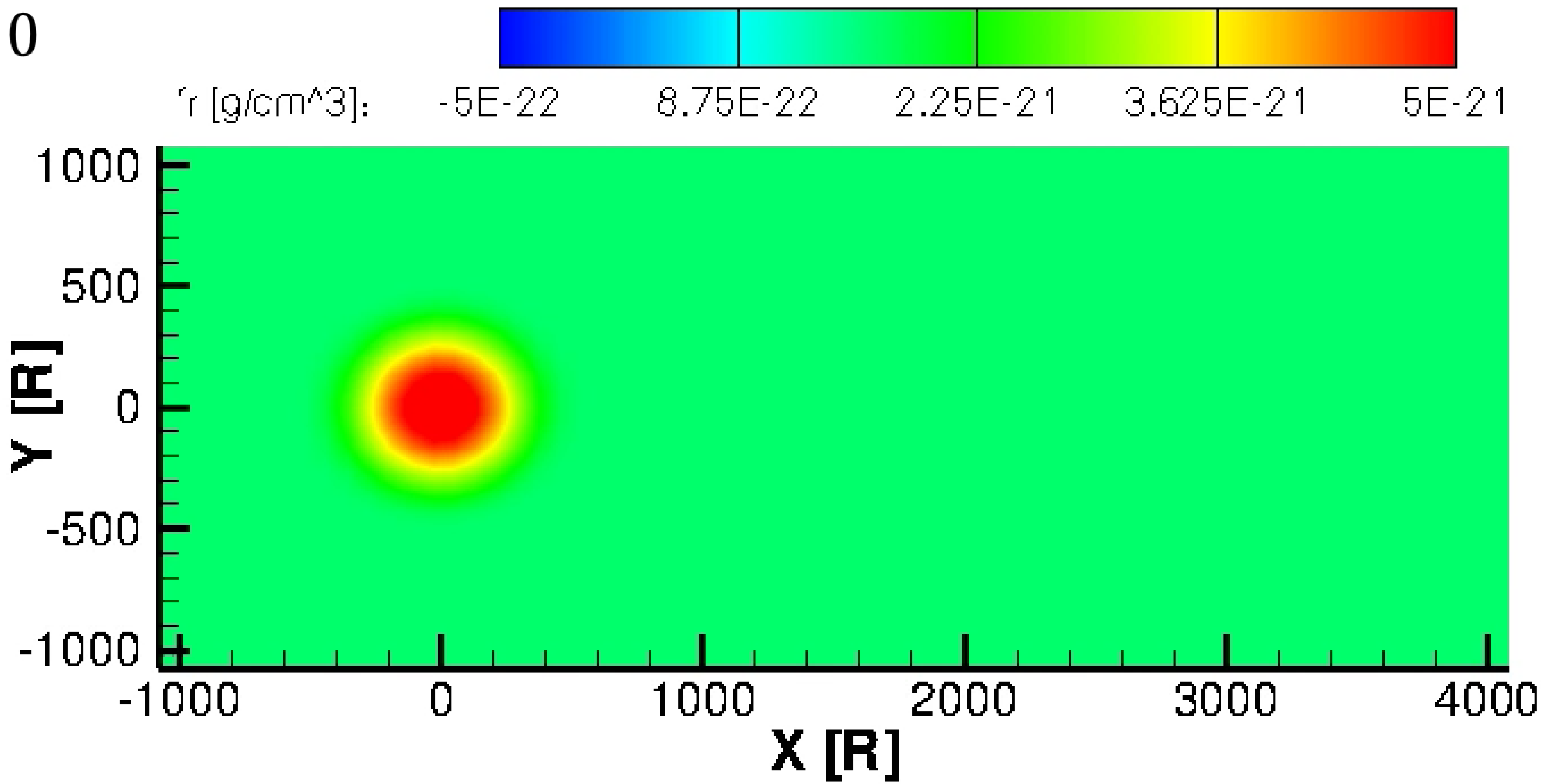
Mathematically, these are hyperbolic partial differential conservation laws:

- **Partial differential equation:** we need to specify initial and boundary conditions
- **Hyperbolic:** for a given initial boundary problem, a solution can be found in any other time and location
  - *Most importantly for solving numerically: information propagates with a characteristic speed*
- **Conservation law:** there is a quantity that is conserved as it is transported

Back to **mass continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

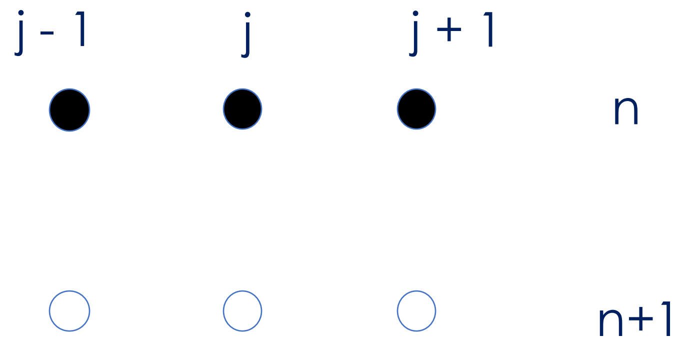
Initial condition: a blob of dense plasma moving in the x direction (to the right) with uniform speed.



How did we simulate it?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \xrightarrow{\text{simplify to 1D:}} \quad \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}(\rho\, u_x) = u_x \frac{\partial}{\partial x}(\rho\,)$$

Discretization over a grid:



Spatial derivative:

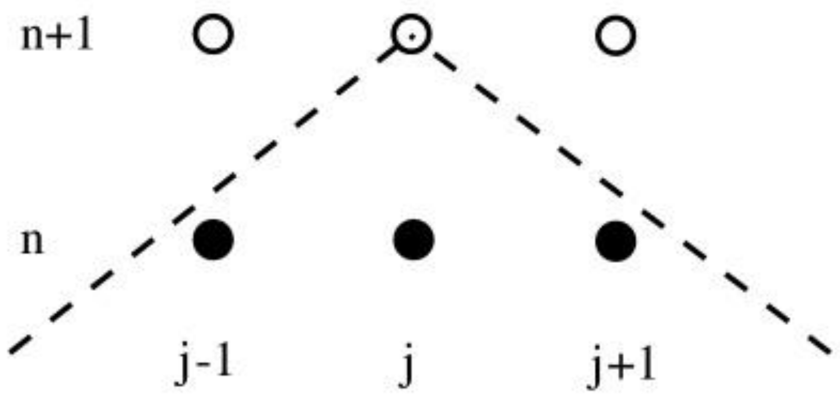
$$\frac{\partial}{\partial x} W(x,t) \xrightarrow{\text{yields}} \frac{W(x_j) \; - W(x_{j-1})}{\Delta x}$$

Temporal derivative:

$$\frac{\partial}{\partial t} W(x,t) \xrightarrow{\text{yields}} \frac{W^{n+1}(x_j) \; - \; W^{n}(x_j)}{\Delta t}$$

Putting it all together

$$W^{n+1}(x_j) = u\Delta t \frac{W^n(x_j) - W^n(x_{j-1})}{\Delta x}$$



The solution at a given location and time depends on information from a limited set of points from the previous time step

MHD Equations

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- **Conservation law:** there is a quantity that is conserved as it is transported

Implications

We are limited in how we chose cell sizes and time steps:

- must allow information to reach the next cell within a time step before we update the solution

Example: a blob moving at 1 km/s
cell size dx = 100 km
time step dt = 10 s

After one time step the blob should move 10 km – it would not reach the next grid point!

$$W^{n+1}(x_j) = u\Delta t \frac{W^n(x_j) - W^n(x_{j-1})}{\Delta x}$$

Updating the solution at j would give non-physical results

To trust or not to trust

Does the solution: $$W^{n+1}(x_j) = u\Delta t\,\frac{W^n(x_j) \;-\; W^n(x_{j-1})}{\Delta x}$$

Reliably mimics: $$\frac{\partial W}{\partial t} = u\frac{\partial}{\partial x}\,W$$

Numerical errors

let's simplify and compare $\frac{W\left(x_{j+1}\right) - W\left(x_j\right)}{dx}$

to: $\frac{\partial W}{\partial x}$

Let's write a Taylor expansion to find the value at the next grid point:

$$W\left(x_{j+1}\right) = W\left(x_j + dx\right)$$

$$W\left(x_{j+1}\right) - W\left(x_j\right) = \left.\frac{\partial W}{\partial x}\right|_{x_j} dx + \frac{1}{2}\left.\frac{\partial^2 W}{\partial x^2}\right|_{x_j} dx^2 + h.o.t$$

$$\frac{W\left(x_{j+1}\right) - W\left(x_j\right)}{dx} = \left.\frac{\partial W}{\partial x}\right|_{x_j} + \frac{1}{2}\left.\frac{\partial^2 W}{\partial x^2}\right|_{x_j} dx + \ldots .$$

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$$\frac{W\left(x_{j+1}\right) - W\left(x_j\right)}{dx} = \boxed{\left.\frac{\partial W}{\partial x}\right|_{x_j}} + \boxed{\frac{1}{2}\left.\frac{\partial^2 W}{\partial x^2}\right|_{x_j} dx + \ldots.}$$

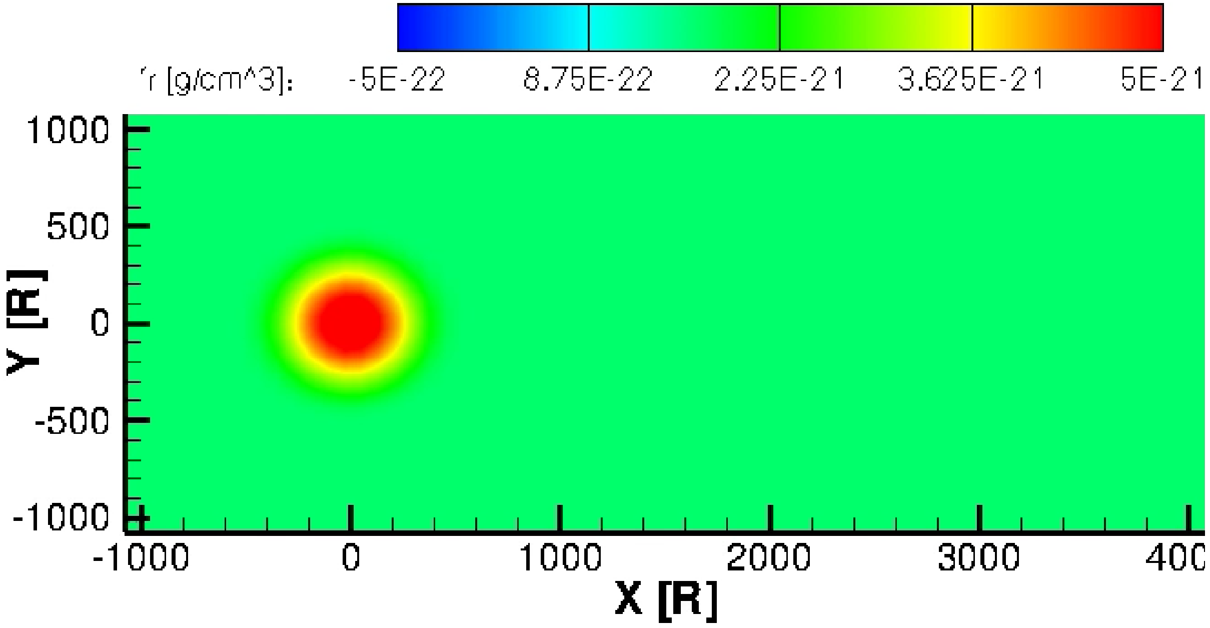
This is what we wanted to simulate

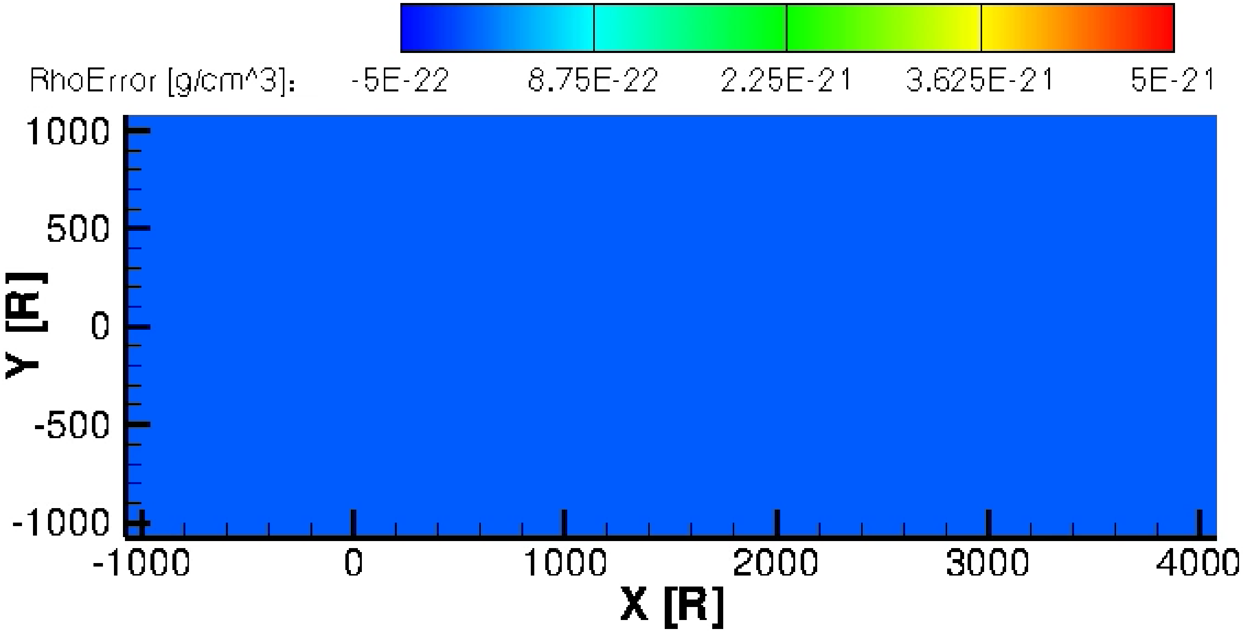
This is the error

"In theory there is no difference between theory and practice, while in practice there is." [Benjamin Brewster, 1881]

Numerical errors

Error compared to an analytical solution





Error analysis

Not so easy to do so for the other MHD equations…

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}\, \mathbf{u} + \left(p + \frac{B^2}{2\mu_0} \right) I - \frac{\mathbf{B}\,\mathbf{B}}{\mu_0} \right] = \rho \mathbf{g}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\,\mathbf{B} - \mathbf{B}\,\mathbf{u}) = 0$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left[\mathbf{u} \left(\varepsilon + p + \frac{B^2}{2\mu_0} \right) - \frac{(\mathbf{u} \cdot \mathbf{B})\,\mathbf{B}}{\mu_0} \right] = \rho \mathbf{g} \cdot \mathbf{u}$$

Error analysis

Every perturbation can be broken into Fourier components, and they will propagate as…

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}\,\mathbf{u} + \left(p + \frac{B^2}{2\mu_0} \right) I - \frac{\mathbf{B}\,\mathbf{B}}{\mu_0} \right] = \rho \mathbf{g}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\,\mathbf{B} - \mathbf{B}\,\mathbf{u}) = 0$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left[\mathbf{u} \left(\varepsilon + p + \frac{B^2}{2\mu_0} \right) - \frac{(\mathbf{u} \cdot \mathbf{B})\,\mathbf{B}}{\mu_0} \right] = \rho \mathbf{g} \cdot \mathbf{u}$$

…MHD waves

Hyperbolic conservation laws

When we derive the MHD wave speeds (linearization, plugging a wave solution, dispersion relations, etc.), we actually find a Jacobean matrix of this system of equations

Hyperbolic differential equations are characterized by this matrix having real eigenvalues – these are the wave speeds.

In the most general sense, every set of coupled hyperbolic equations has characteristic waves speeds – this is how physical information propagates!

Examples

Euler equation (fluid dynamics): acoustic speeds

E&M wave equation: speed of light

MHD equations: Alfven, fast and slow magnetosonic

Check yourself: there are always as many speeds as there are equations!

For fluid equations, one speed is always the flow speed: whatever structure we have, it is transported with the flow (see blob example).

Practical Implications: **time stepping**

$$\Delta t = C\left(\frac{c_x + |u_x|}{\Delta x} + \frac{c_y + |u_y|}{\Delta y} + \frac{c_z + |u_z|}{\Delta z}\right)^{-1}$$

C < 1

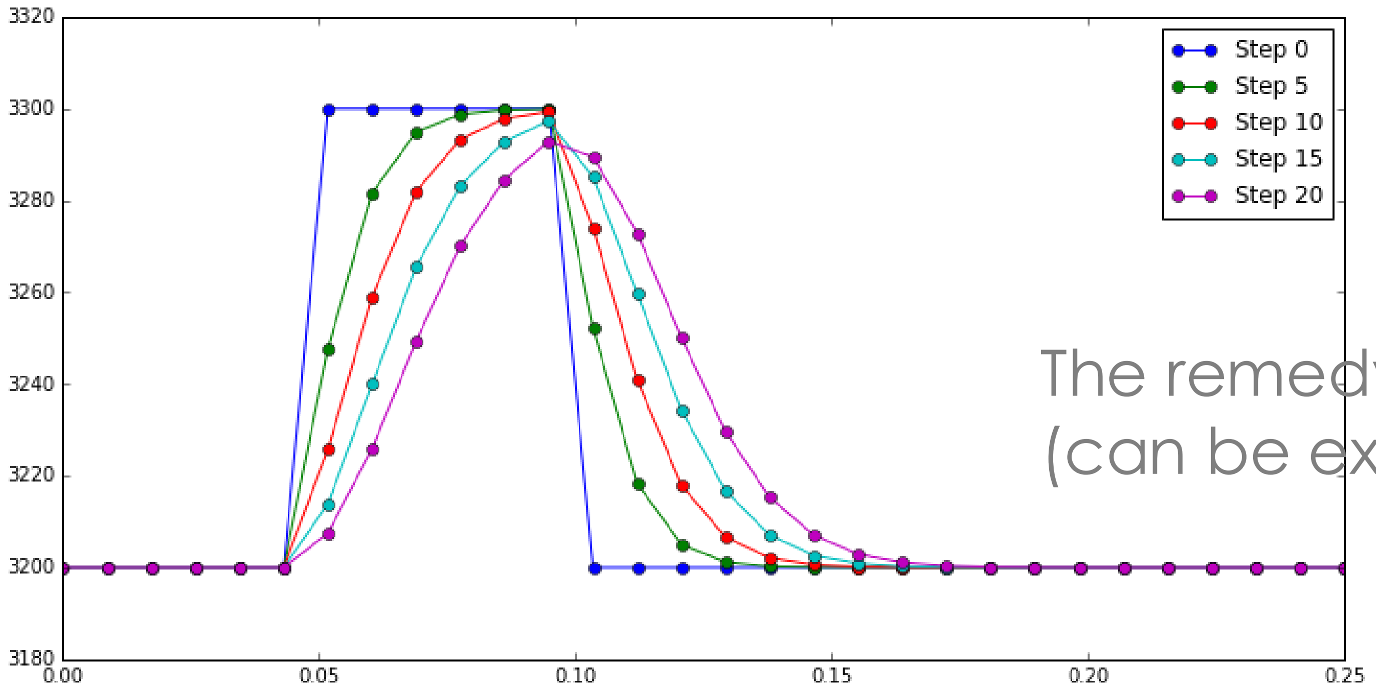
Have to take into account all the wave speeds!

This can make simulations very slow

Implications: **numerical diffusion**

$$\frac{W\,(x_{j+1})\; - \;W\,(x_j)}{dx} = \left.\frac{\partial W}{\partial x}\right|_{x_j} + \boxed{\left.\frac{1}{2}\,\frac{\partial^2 W}{\partial x^2}\right|_{x_j} dx} + \;\ldots.$$

diffusion-like term



The remedy: decrease dx
(can be expensive)

Implications: **div B** control

According to Maxwell's equations, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$

Discretization errors will generally violate this equality

The residual $\boldsymbol{\nabla} \cdot \boldsymbol{B}$ has to be "*evicted*"

$$
\begin{aligned}
\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot \left(\mathbf{u}\rho\mathbf{u} - \frac{\mathbf{BB}}{\mu_0}\right) + \nabla\left(p + \frac{\mathbf{B}^2}{2\mu_0}\right) &= \boxed{-\frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B}} \\
\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{uB} - \mathbf{Bu}) &= \boxed{-(\nabla \cdot \mathbf{B})\mathbf{u}} \\
\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left[\mathbf{u}\left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma - 1}p + \frac{B^2}{\mu_0}\right) - \frac{(\mathbf{u}\cdot\mathbf{B})\,\mathbf{B}}{\mu_0}\right] &= \boxed{-\frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B}\cdot\mathbf{u}}
\end{aligned}
$$

Word of caution:

In the real world of computational physics (and numerical simulations in general), we don't have analytical solutions to compare to.

If we did, we wouldn't need simulations…

Hierarchy of descriptions of plasmas

Individual particles

(reduce number of particles)

particle in cell (PIC)

(distribution function instead of individual particles)

kinetic (6D)

kinetic (reduced)

(electrons treated as a fluid, protons as particles)

hybrid fluid-PIC

higher moment fluid

extended MHD

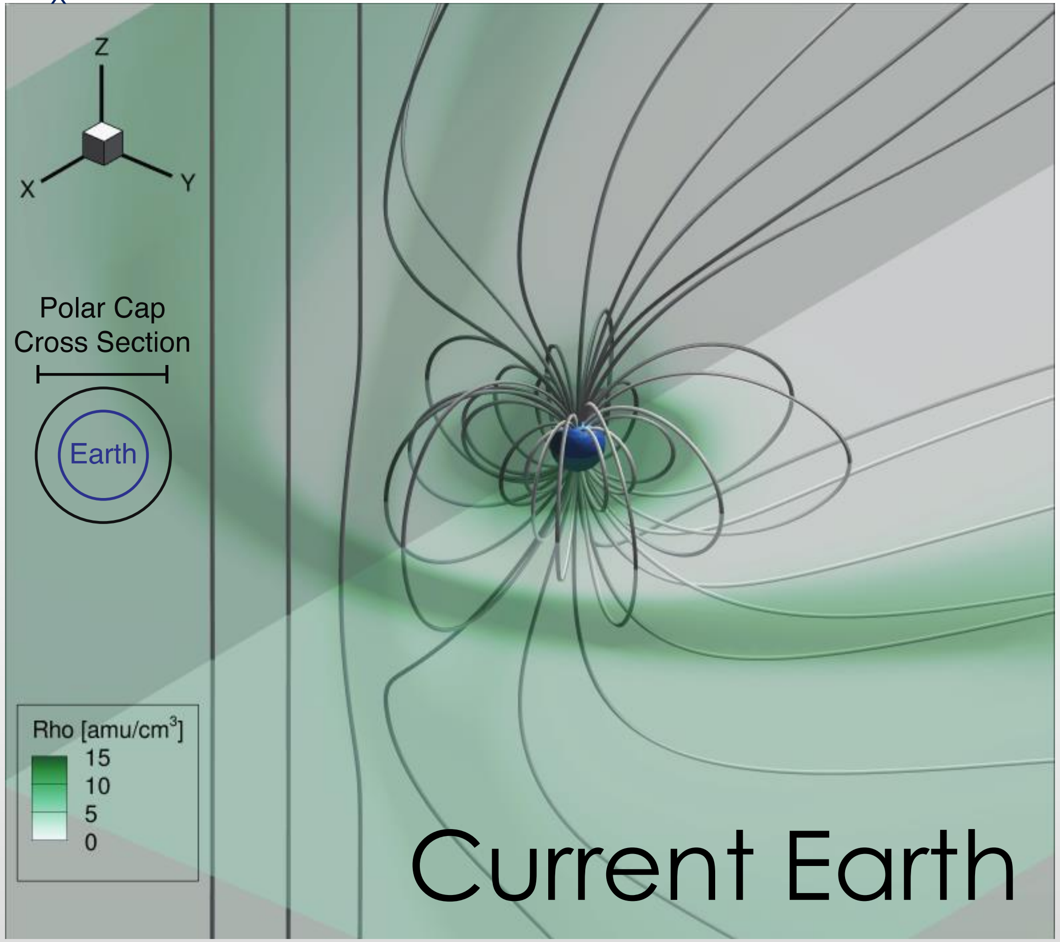
ideal MHD

more complex physics, still a fluid description

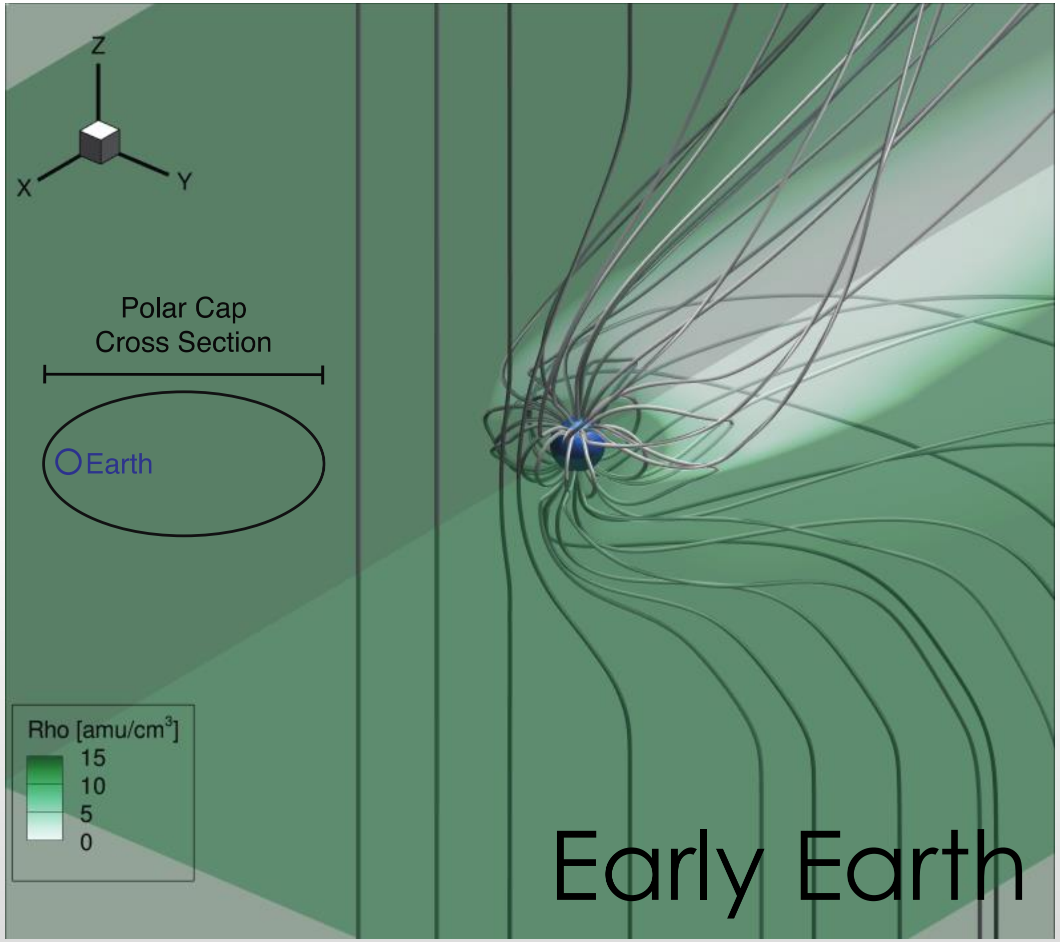
Fluid

Ideal MDH simulations

IMF: B_z = -6 *nT*,
v_x = -400 km/s , n = 5 cm^{-3}.



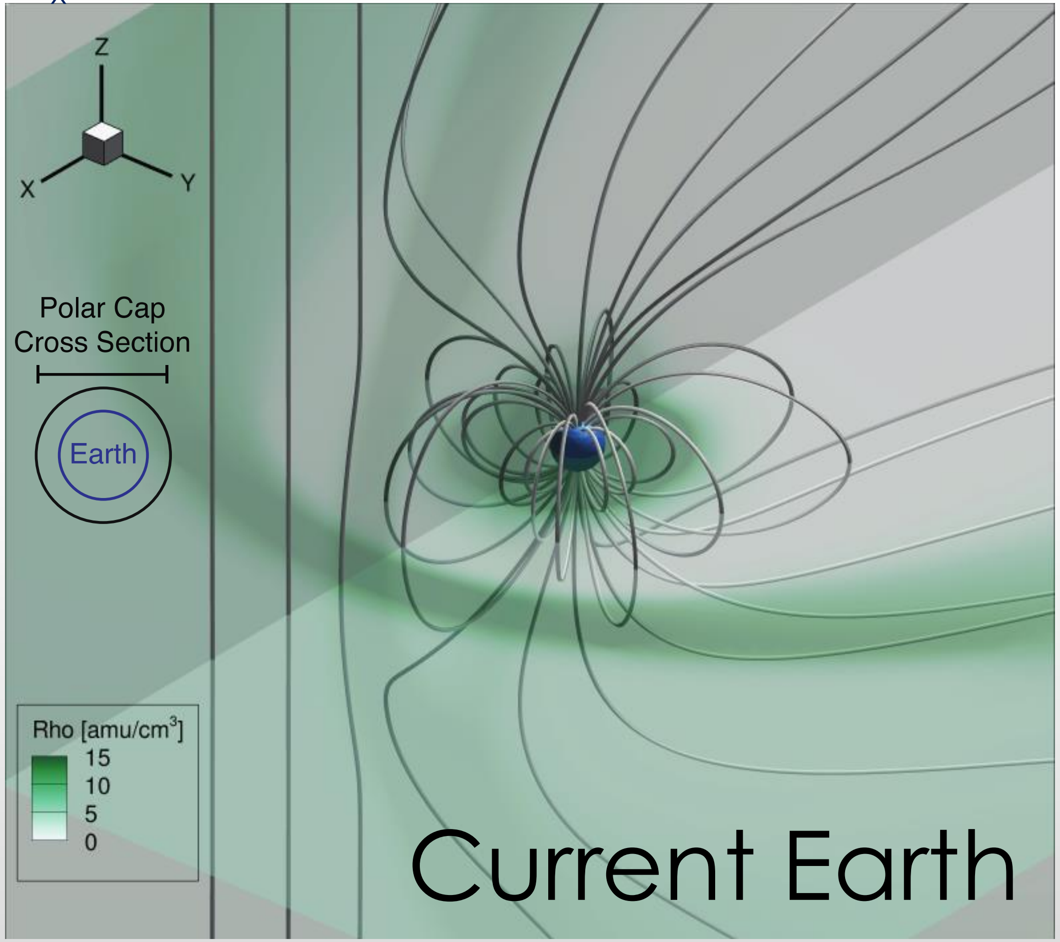
IMF: B_z = -9 *nT*,
v_x = -400 km/s , **n = 26 cm^{-3}, M = 0.1M_E**




LET'S TALK

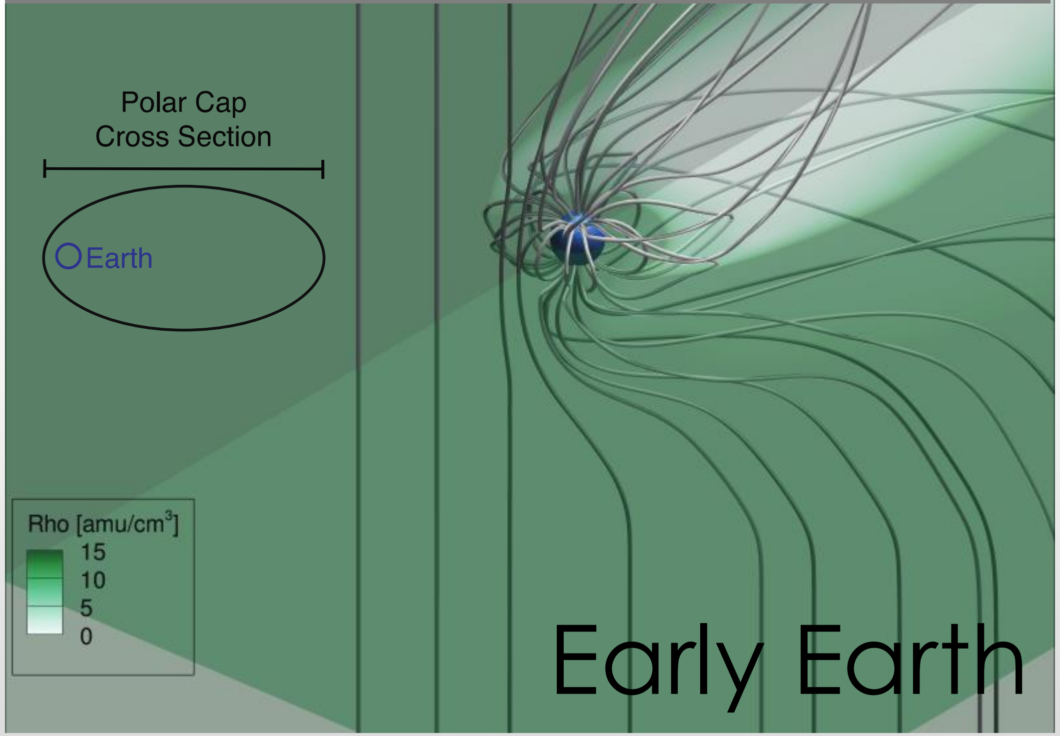
Ideal MDH simulations

IMF: $B_z = -6\ nT$,
v_x = -400 km/s , n = 5 cm^{-3}.



IMF: $B_z = -9\ nT$,
v_x = -400 km/s , **n = 26 cm^{-3}, M = 0.1M_E**

Which of the above parameters are the dominant factors in shaping the size of the polar cap?



Hierarchy of descriptions of plasmas

	Physical description	Computational feasibility
"All particles and fields" Solve the EOM for all particles in given **E**, **B** Calculate resulting charge density and currents Calculate E, B	complete	impossible
Macro particles/statistical Sample the distribution function Solve for macro particles that represent many particles with similar states	most physical processes captured	expensive
Kinetic Discretize the distribution function over a grid in phase space. Solve Boltzmann/Vlasov equation (with or without collision terms)	many physical processes captured	expensive – need to solve 6D equations

Hierarchy of descriptions of plasmas

	Physical description	Computational feasibility
“Hybrid” Solve the EOM for ions, fluid equations for electrons Calculate resulting charge density and currents	cannot resolve electron scale dynamics	Speed depends on number of macroparticles. Solution can be noisy
Higher moment fluid equations Solve fluid(s) equations with higher-order closure: pressure is a 2nd order tensor (9 elements), heat flux is a 3rd order tensor, etc.	cannot resolve particle dynamics	Complex relationships, many coupled equations
Hall MHD Incorporate full form of Ohm’s law	higher-order moments neglected, smooth solution	

Hierarchy of descriptions of plasmas

	Physical description	Computational feasibility
Extended MHD (many combinations) Extend state vector to include anisotropic pressure (represented by diagonal tensor with 3 elements), separate electron and ion pressures, heat conduction vector (not 3D tensor), radiative cooling, resistivity, etc.	imposes frozen-in regime, cannot describe reconnection	second-order derivatives may make the problem stiff, no exact Riemann solver
Ideal MHD Solve fluid equations with scalar pressure and assuming thermal equilibrium	imposes frozen-in regime, cannot describe reconnection	many efficient, scalable codes exist