Creation and destruction of magnetic fields

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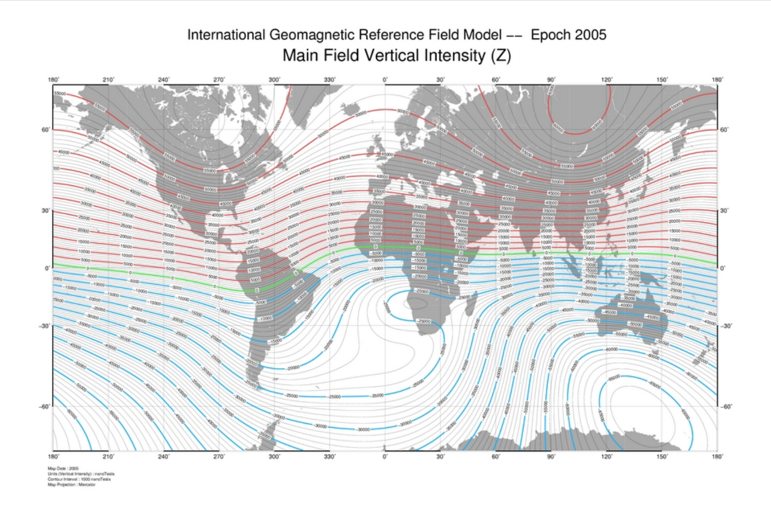
Scope of this lecture

- Processes of magnetic field generation and destruction in turbulent plasma flows
- Introduction to general concepts of dynamo theory
- Outline
  - Intro: Magnetic fields in the Universe
  - MHD, induction equation
  - Some general remarks and definitions regarding dynamos
  - Small-scale dynamos
  - Large-scale dynamos (mean field theory)
    - Kinematic theory
    - Characterization of possible dynamos
    - Non-kinematic effects
  - Concluding remarks

Magnetic fields in the Universe

- Earth
  - Field strength ~ 0.5G
  - Magnetic field present for $\sim 3.5 \cdot 10^9$ years, much longer than Ohmic decay time ($\sim 10^4$ years)
  - Strong variability on shorter time scales (10^3 years)
- Mercury, Ganymede, (Io), Jupiter, Saturn, Uranus, Neptune have large scale fields
- Sun
  - Magnetic fields from smallest observable scales to size of sun
  - 22 year cycle of large-scale field
  - Ohmic decay time $\sim 10^9$ years (in absence of turbulence)
- Other stars
  - Stars with outer convection zone: similar to sun
  - Stars with outer radiation zone: primordial fields, field generation in convective core
- Galaxies
  - Field strength $\sim \mu$G
  - Field structure coupled to observed matter distribution

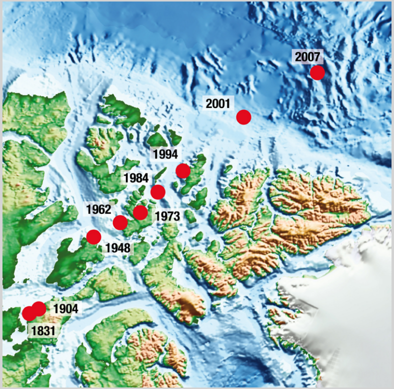
Mostly dipolar field structure (currently)



Credit: NOAA NGDC

Geomagnetism

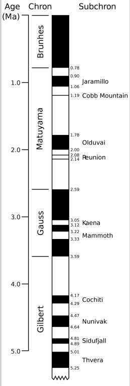
Short-term variation on scales of hundreds of years

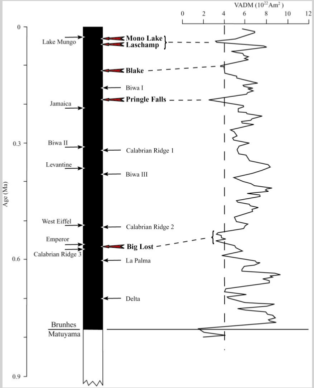


Credit: Arnaud Chulliat (Institut de Physique du Globe de Paris)

- Independent movement of the poles
- South and North pole are in general not opposite to each other (higher multipoles)
- Movements up to 40 km/year (~ 1 mm/sec)

Long-term variation on scales of thousands to millions of years (deduced from volcanic rocks and sediments)

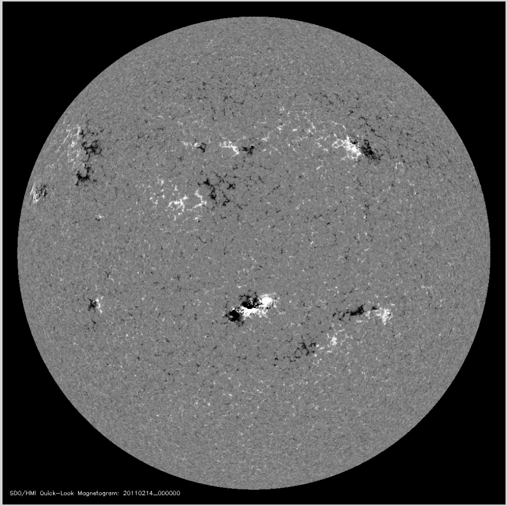




Credit: US Geological Survey

- Mostly random changes of polarity
- A given polarity for $\sim 100,000$ years
- Fast switches ~ 1000 years
- Strong variation of dipole moment and failed reversals

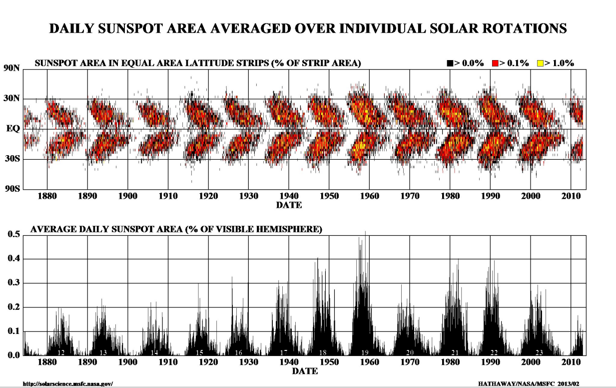
Solar Magnetism



Full disk magnetogram SDO/HMI

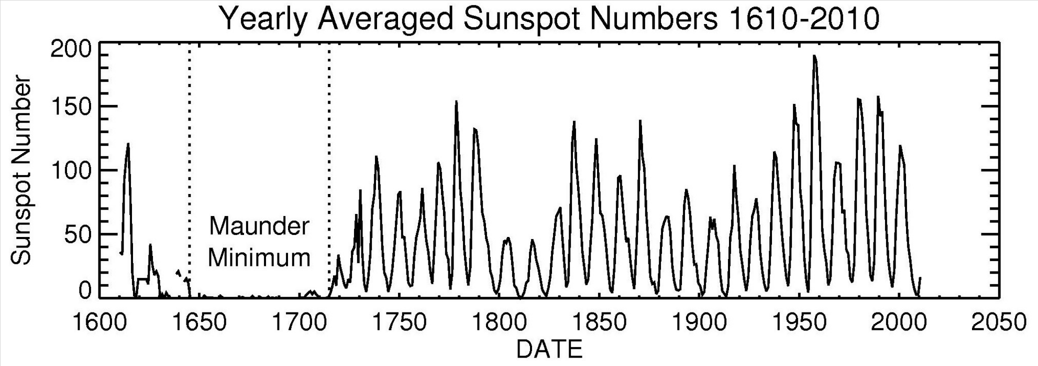
- Up to 4kG (sunspot umbra) field in solar photosphere
- Structured over the full range of observable scales from 100 km to size of Sun
- Large-scale field shows symmetries with respect to equator and periodic reversals
- Small-scale field appears to be mostly independent from large-scale field

Solar Magnetism



- Large-scale field exhibits $\sim$ 22 year magnetic cycle
- 11 year cycle present in large-scale flow variations (meridional flow and differential rotation)

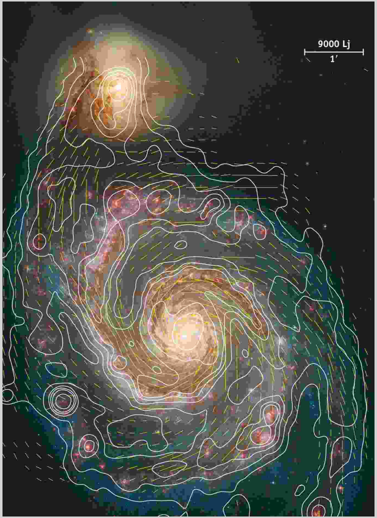
Solar Magnetism



Credit: NASA

- Cycle interrupted by grand minima with duration of up to 100 years
- Similar overall activity has been present for past $\sim 100,000$ years (tree ring and ice core records of cosmogenic isotopes: C-14 and Be-10).

Galactic magnetism



M51, Credit: MPI for Radioastronomy, Germany

- Magnetic field derived from polarization of radio emission
- μG field strength
- Magnetic field follows spiral structure to some extent
- Optically thin dynamo - Dynamo region can be observed!

Magnetic fields in the Universe

- Objects from size of a planet to galaxy clusters have large-scale ($\sim$ size of object) magnetic fields
- Physical properties of object differ substantially
  - 1,000 km to 100,000 LJ
  - liquid iron to partially ionized plasma
  - spherical to disk-shaped
  - varying influence of rotation (but all of them are rotating)
  - $R_m \sim 10^3 \ldots 10^{18}$
  - ....
- Is there a common origin of magnetic field in these objects?
- Can we understand this on basis of MHD?

MHD equations

The basic framework for understanding the dynamics of a magnetized fluid are the MHD equations. In their most simple form they are applicable under the following conditions:

- Validity of continuum approximation (enough particles to define averages)
- Strong collisional coupling: validity of single fluid approximations, isotropic (scalar) gas pressure
- Non-relativistic motions, low frequencies, high electrical conductivity

They combine a fluid description in terms of the Navier-Stokes equations with the non-relativistic Maxwell equations as well as Ohm's Law.

Kinematic approach

- Solving the 3D MHD equations is not always feasible
- Semi-analytical approach preferred for understanding fundamental properties of dynamos
- Evaluate turbulent induction effects based on induction equation for a given velocity field
    - Velocity field assumed to be given as 'background' turbulence, Lorentz-force feedback neglected (sufficiently weak magnetic field)
    - What correlations of a turbulent velocity field are required for dynamo (large-scale) action?
    - Theory of onset of dynamo action, but not for non-linear saturation

- More detailed discussion of induction equation

Ohm's law

Equation of motion for drift velocity $\boldsymbol{v}_d$ of electrons

$$n_e m_e \left(\frac{\partial v_d}{\partial t} + \frac{v_d}{\tau_{ei}} \right) = n_e q_e (\boldsymbol{E} + \boldsymbol{v}_d \times \boldsymbol{B}) - \boldsymbol{\nabla} p_e$$

τ_{ei}: collision time between electrons and ions

n_e: electron density

q_e: electron charge

m_e: electron mass

p_e: electron pressure

With the electric current: $\boldsymbol{j} = n_e \, q_e \, \boldsymbol{v}_d$ this gives the generalized Ohm's law:

$$\frac{\partial \boldsymbol{j}}{\partial t} + \frac{\boldsymbol{j}}{\tau_{ei}} = \frac{n_e q_e^2}{m_e} \boldsymbol{E} + \frac{q_e}{m_e} \boldsymbol{j} \times \boldsymbol{B} - \frac{q_e}{m_e} \boldsymbol{\nabla} p_e$$

Simplifications:

- $\tau_{ei} \, \omega_L \ll 1$, $\omega_L = eB/m_e$: Larmor frequency
- neglect $\boldsymbol{\nabla} p_e$
- low frequencies (no plasma oscillations)

Ohm's law

Simplified Ohm's law

$$\boldsymbol{j} = \sigma \boldsymbol{E}$$

with the plasma conductivity

$$\sigma = \frac{\tau_{ei} n_e q_e^2}{m_e}$$

The Ohm's law we derived so far is only valid in the co-moving frame of the plasma. Under the assumption of non-relativistic motions this transforms in the laboratory frame to

$$\boldsymbol{j} = \sigma \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right)$$

Induction equation

Using Ampere's law $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$ yields for the electric field in the laboratory frame

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \frac{1}{\mu_0 \sigma} \boldsymbol{\nabla} \times \boldsymbol{B}$$

leading to the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B} - \eta \boldsymbol{\nabla} \times \boldsymbol{B})$$

with the magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma} .$$

Advection, diffusion, magnetic Reynolds number

L: typical length scale U: typical velocity scale L/U: time unit

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left(\boldsymbol{v} \times \boldsymbol{B} - \frac{1}{R_m} \boldsymbol{\nabla} \times \boldsymbol{B} \right)$$

with the magnetic Reynolds number

$$R_m = \frac{U\,L}{\eta}\ .$$

Advection, diffusion, magnetic Reynolds number

$R_m \ll 1$: diffusion dominated regime

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \Delta \boldsymbol{B} \ .$$

Only decaying solutions with decay (diffusion) time scale

$$\tau_{\mathrm{d}} \sim \frac{L^2}{\eta}$$

Object	$\eta[\mathrm{m}^2/\mathrm{s}]$	$L[\mathrm{m}]$	$U[\mathrm{m/s}]$	R_m	τ_{d}
earth (outer core)	2	10^6	10^{-3}	300	10^4 years
sun (plasma conductivity)	1	10^8	100	10^{10}	10^9 years
sun (turbulent conductivity)	10^8	10^8	100	100	3 years
liquid sodium lab experiment	0.1	1	10	100	10 s

Advection, diffusion, magnetic Reynolds number

$R_m \gg 1$ advection dominated regime (ideal MHD)

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B})$$

Equivalent expression

$$\frac{\partial \boldsymbol{B}}{\partial t} = -(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{B}\,\boldsymbol{\nabla} \cdot \boldsymbol{v}$$

- advection of magnetic field
- amplification by shear (stretching of field lines)
- amplification through compression

Advection, diffusion, magnetic Reynolds number

Incompressible fluid ($\nabla \cdot \boldsymbol{v} = 0$):

$$\frac{d\boldsymbol{B}}{dt} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}$$

Velocity shear in the direction of $\boldsymbol{B}$ plays key role. Mathematically similar equation for compressible fluid (Walen equation):

$$\frac{d}{dt}\frac{\boldsymbol{B}}{\varrho} = \left(\frac{\boldsymbol{B}}{\varrho} \cdot \nabla\right)\boldsymbol{v}$$

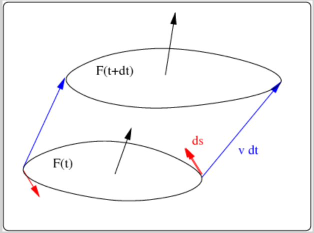
Vertical flux transport in statified medium:

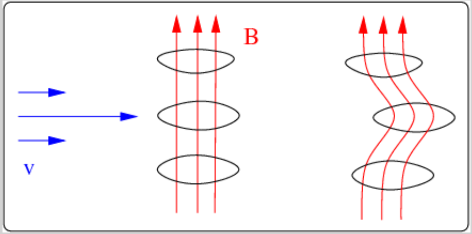
- $B \sim \varrho$ no expansion in direction of $\boldsymbol{B}$
- $B \sim \varrho^{2/3}$ isotropic expansion
- $B \sim \varrho^{1/2}$ 2D expansion in plane containing $\boldsymbol{B}$
- $B = const.$ only expansion in direction of $\boldsymbol{B}$

Alfven's theorem

Let Φ be the magnetic flux through a surface F with the property that its boundary ∂F is moving with the fluid:

$$\Phi = \int_F \boldsymbol{B} \cdot d\boldsymbol{f} \longrightarrow \frac{d\Phi}{dt} = 0$$





- Flux is 'frozen' into the fluid
- Field lines 'move' with plasma

Dynamos: Motivation

- For $\boldsymbol{v} = 0$ magnetic field decays on timescale $\tau_d \sim L^2/\eta$
- Earth and other planets:
  - Evidence for magnetic field on earth for $3.5 \cdot 10^9$ years while $\tau_d \sim 10^4$ years
  - Permanent rock magnetism not possible since $T > T_{\mathrm{Curie}}$ and field highly variable $\longrightarrow$ field must be maintained by active process
- Sun and other stars:
  - Evidence for solar magnetic field for $\sim 300\,000$ years (^{10}Be)
  - Most solar-like stars show magnetic activity (details depend on stellar type and rotation)
  - Indirect evidence for stellar magnetic fields over life time of stars
  - But $\tau_d \sim 10^9$ years!
  - Primordial field could have survived in radiative interior of sun, but convection zone has much shorter diffusion time scale ~ 10 years (turbulent diffusivity)
  - Variability on time-scales $\ll \tau_d$.

Mathematical definition of dynamo

S bounded volume with the surface ∂S, $\boldsymbol{B}$ maintained by currents contained within S, $B \sim r^{-3}$ asymptotically,

$$\begin{aligned} \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B}) && \text{in } S \\ \nabla \times \boldsymbol{B} &= 0 && \text{outside } S \\ [\boldsymbol{B}] &= 0 && \text{across } \partial S \\ \nabla \cdot \boldsymbol{B} &= 0 \end{aligned}$$

$\boldsymbol{v} = 0$ outside S, $\boldsymbol{n} \cdot \boldsymbol{v} = 0$ on ∂S and

$$E_{\mathrm{kin}} = \int_S \frac{1}{2} \varrho \boldsymbol{v}^2 \, dV \leq E_{\mathrm{max}} \quad \forall \, t$$

$\boldsymbol{v}$ is a dynamo if an initial condition $\boldsymbol{B} = \boldsymbol{B}_0$ exists so that

$$E_{\mathrm{mag}} = \int_{-\infty}^{\infty} \frac{1}{2\mu_0} \boldsymbol{B}^2 \, dV \geq E_{\mathrm{min}} \quad \forall \, t$$

Mathematical definition of dynamo

- Is this dynamo different from those found in powerplants?
  - Both have conducting material and relative motions (rotor/stator in powerplant vs. shear flows)
- Difference mostly in one detail:
  - Dynamos in powerplants have wires (very inhomogeneous conductivity), i.e. the electric currents are strictly controlled
  - Mathematically the system is formulated in terms of currents
  - A short circuit is a major desaster!
  - For astrophysical dynamos we consider homogeneous conductivity, i.e. current can flow anywhere
  - Mathematically the system is formulated in terms of $\boldsymbol{B}$ ($\boldsymbol{j}$ is eliminated from equations whenever possible).
  - A short circuit is the normal mode of operation!
- Homogeneous vs. inhomogeneous dynamos

Large-scale/small-scale dynamos

Decompose the magnetic field into large-scale part and small-scale part (energy carrying scale of turbulence) $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{B}'$:

$$E_{\mathrm{mag}} = \int \frac{1}{2\mu_0} \overline{\boldsymbol{B}}^2 \, dV + \int \frac{1}{2\mu_0} \overline{\boldsymbol{B}'^2} \, dV \; .$$

- Small-scale dynamo: $\overline{\boldsymbol{B}}^2 \ll \overline{\boldsymbol{B}'^2}$
- Large-scale dynamo: $\overline{\boldsymbol{B}}^2 \geq \overline{\boldsymbol{B}'^2}$

Almost all turbulent (chaotic) velocity fields are small-scale dynamos for sufficiently large R_m, large-scale dynamos require additional large scale symmetries (see second half of this lecture)

What means large/small scale?

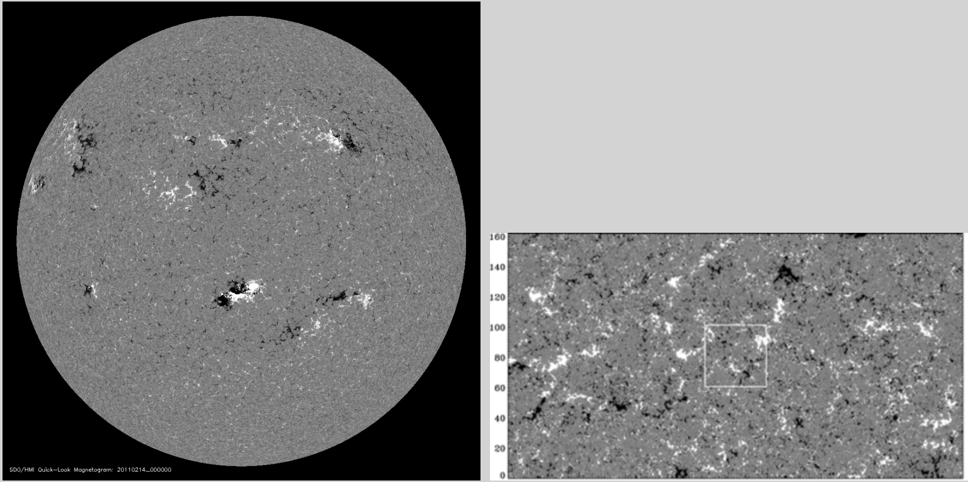


Figure: Full disk magnetogram SDO/HMI, Hinode magnetogram

Small-scale dynamo (SSD) action

Lagrangian particle paths:

$$\frac{d\boldsymbol{x}_1}{dt} = \boldsymbol{v}(\boldsymbol{x}_1, t) \qquad \frac{d\boldsymbol{x}_2}{dt} = \boldsymbol{v}(\boldsymbol{x}_2, t)$$

Consider small separations:

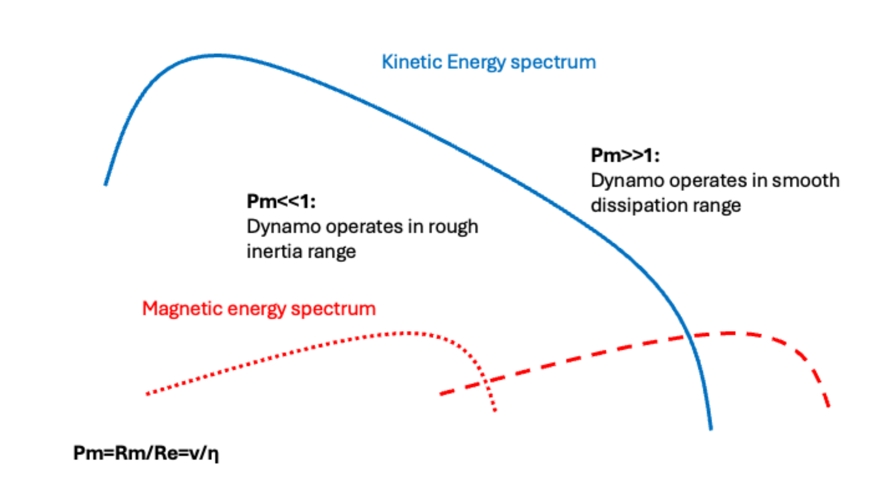
$$\boldsymbol{\delta} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \qquad \frac{d\boldsymbol{\delta}}{dt} = (\boldsymbol{\delta} \cdot \boldsymbol{\nabla})\boldsymbol{v}$$

Chaotic flows have exponentially growing solutions. Due to mathematical simularity the equation:

$$\frac{d}{dt}\frac{\boldsymbol{B}}{\varrho} = \left(\frac{\boldsymbol{B}}{\varrho} \cdot \boldsymbol{\nabla}\right)\boldsymbol{v}$$

has exponentially growing solutions, too. We neglected here η, exponentially growing solutions require $R_m > \mathcal{O}(100)$.

The magnetic Prandtl number (P_{m}) challenge



The magnetic Prandtl number (P_m) challenge

Stellar convection zone have generally small magnetic Prandtl numbers

$$
\begin{aligned}
R_\mathrm{e} &\gg R_\mathrm{m} \gg 0 \\
P_\mathrm{m} &= \frac{R_\mathrm{m}}{R_\mathrm{e}} \ll 1
\end{aligned}
$$

Typical solar P_m values are 10^{-2} (base of CZ) to 10^{-5} (Photosphere). Early SSD simulations used $P_\mathrm{m} > 1$ and found that the critical R_m was increasing as P_m was lowered. Do SSDs exist in the limit of small P_m?

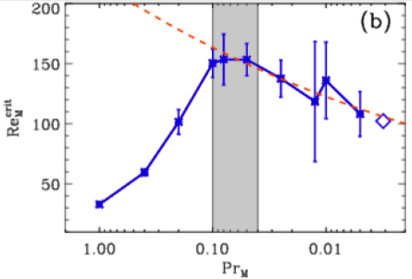
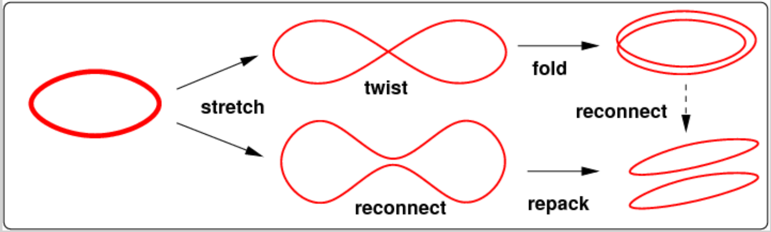


Figure: From Warnecke et al. (2023)

Large-scale/small-scale dynamos



- Amplification through field line stretching
- Twist-fold required to repack field into original volume
- Twist-fold requires 3D - there are no dynamos is 2D!
- Magnetic diffusivity allows for change of topology

Slow/fast dynamos

Influence of magnetic diffusivity on growth rate

- Fast dynamo: growth rate independent of R_m (stretch-twist-fold mechanism)
- Slow dynamo: growth rate limited by resistivity (stretch-reconnect-repack)

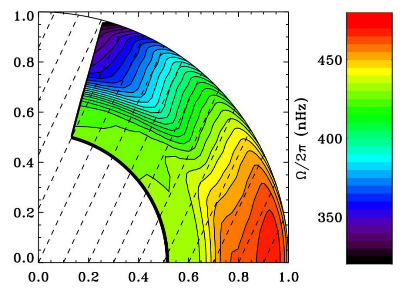
- Fast dynamos relevant for most astrophysical objects since $R_m \gg 1$
- Dynamos including (resistive) reconnection steps can be fast provided the reconnection is fast

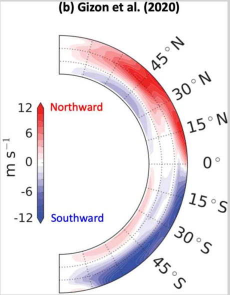
Differential rotation and meridional flow

Induction effects of axisymmetric flows on axisymmetric field:

$$
\begin{aligned}
\boldsymbol{B} &= B\boldsymbol{e_\Phi} + \boldsymbol{\nabla} \times (A\boldsymbol{e_\Phi}) \\
\boldsymbol{v} &= v_r\boldsymbol{e_r} + v_\theta\boldsymbol{e_\theta} + \Omega\, r \sin\theta\, \boldsymbol{e_\Phi}
\end{aligned}
$$

Differential rotation most dominant shear flow in stellar convection zones:





Meridional flow by-product of DR, observed as poleward surface flow in case of the sun

Differential rotation and meridional flow

Spherical geometry:

$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r B) + \frac{\partial}{\partial \theta}(v_\theta B)\right) =$$

$$r \sin \, \boldsymbol{B}_p \cdot \boldsymbol{\nabla}\Omega + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) B$$

$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\, \boldsymbol{v}_p \cdot \boldsymbol{\nabla}(r\sin\theta A) = \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) A$$

- Meridional flow: Independent advection of poloidal and toroidal field
- Differential rotation: Source for toroidal field (if poloidal field not zero)
- Diffusion: Sink for poloidal and toroidal field
- No term capable of maintaining poloidal field against Ohmic decay!

Differential rotation and meridional flow

- Weak poloidal seed field can lead to significant field amplification
- No source term for poloidal field
- Decay of poloidal field on resistive time scale
- Ultimate decay of toroidal field
- Not a dynamo!
- What is needed?

- Source for poloidal field

Cowling's anti-dynamo theorem

A stationary axisymmetric magnetic field with currents limited to a finite volume in space cannot be maintained by a velocity field with finite amplitude.

If dynamos exist, they require more complex, non-axisymmetric magnetic fields!

Large-scale dynamo theory

Some history:

- 1919 Sir Joeseph Larmor: Solar magnetic field maintained by motions of conducting fluid?
- 1937 Cowling's anti-dynamo theorem and many others
- 1955 Parker: decomposition of field in axisymmetric and non-axisymmetric parts, average over induction effects of non-axisymmetric field
- 1964 Braginskii, Steenbeck, Krause: Mathematical frame work of mean field theory developed
- last 2 decades 3D dynamo simulations

Reynolds rules

We need to define an averaging procedure to define the mean and the fluctuating field. For any function f and g decomposed as $f = \overline{f} + f'$ and $g = \overline{g} + g'$ we require that the Reynolds rules apply

$$
\begin{aligned}
\overline{\overline{f}} &= \overline{f} \longrightarrow \overline{f'} = 0 \\
\overline{f+g} &= \overline{f} + \overline{g} \\
\overline{f\overline{g}} &= \overline{f}\,\overline{g} \longrightarrow \overline{f'\overline{g}} = 0 \\
\overline{\partial f/\partial x_i} &= \partial \overline{f}/\partial x_i \\
\overline{\partial f/\partial t} &= \partial \overline{f}/\partial t\,.
\end{aligned}
$$

Examples:

- Longitudinal average (mean = axisymmetric component)
- Ensemble average (mean = average over several realizations of chaotic system)

Meanfield induction equation

Average of induction equation:

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left(\overline{\boldsymbol{v}' \times \boldsymbol{B}'} + \overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}} - \eta \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}\right)$$

New term resulting from small-scale effects:

$$\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$$

Fluctuating part of induction equation:

$$\left(\frac{\partial}{\partial t} - \eta \Delta\right) \boldsymbol{B}' - \boldsymbol{\nabla} \times (\overline{\boldsymbol{v}} \times \boldsymbol{B}') = \boldsymbol{\nabla} \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}} + \boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}\right)$$

Kinematic approach: $\boldsymbol{v}'$ assumed to be given

- Solve for $\boldsymbol{B}'$, compute $\overline{\boldsymbol{v}' \times \boldsymbol{B}'}$ and solve for $\overline{\boldsymbol{B}}$
- Term $\boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$ leading to higher order correlations (closure problem)

Mean field expansion of turbulent induction effects

Exact expressions for $\overline{\boldsymbol{\mathcal{E}}}$ exist only under strong simplifying assumptions (see homework assignment).

In general $\overline{\boldsymbol{\mathcal{E}}}$ is a linear functional of $\overline{\boldsymbol{B}}$:

$$\overline{\mathcal{E}}_i(\boldsymbol{x},t) = \int_{-\infty}^{\infty} d^3x' \int_{-\infty}^{t} dt' \, \mathcal{K}_{ij}(\boldsymbol{x},t,\boldsymbol{x}',t') \, \overline{B}_j(\boldsymbol{x}',t') \, .$$

Can be simplified if a sufficient scale separation is present:

- $l_c \ll L$
- $\tau_c \ll \tau_L$

Leading terms of expansion:

$$\overline{\mathcal{E}}_i = a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k}$$

In stellar convection zones scale separation also only marginally justified (continuous turbulence spectrum)!

Symmetry constraints

Decomposing a_{ij} and $\partial \overline{B}_j / \partial x_k$ into symmetric and antisymmetric components:

$$
\begin{aligned}
a_{ij} &= \underbrace{\frac{1}{2}\left(a_{ij} + a_{ji}\right)}_{\alpha_{ij}} + \underbrace{\frac{1}{2}\left(a_{ij} - a_{ji}\right)}_{-\varepsilon_{ijk}\gamma_k} \\
\frac{\partial \overline{B}_j}{\partial x_k} &= \frac{1}{2}\left(\frac{\partial \overline{B}_j}{\partial x_k} + \frac{\partial \overline{B}_k}{\partial x_j}\right) + \underbrace{\frac{1}{2}\left(\frac{\partial \overline{B}_j}{\partial x_k} - \frac{\partial \overline{B}_k}{\partial x_j}\right)}_{-\frac{1}{2}\varepsilon_{jkl}(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_l}
\end{aligned}
$$

Leads to:

$$
\overline{\mathcal{E}}_i = \alpha_{ij}\overline{B}_j + \varepsilon_{ikj}\gamma_k\overline{B}_j - \underbrace{\frac{1}{2}b_{ijk}\varepsilon_{jkl}}_{\beta_{il}-\varepsilon_{ilm}\delta_m}(\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})_l + \ldots
$$

Symmetry constraints

Overall result:

$$\overline{\boldsymbol{\mathcal{E}}} = \boldsymbol{\alpha}\overline{\boldsymbol{B}} + \boldsymbol{\gamma} \times \overline{\boldsymbol{B}} - \boldsymbol{\beta}\,\boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \boldsymbol{\delta} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}) + \ldots$$

With:

$$\begin{aligned} \alpha_{ij} &= \frac{1}{2}\left(a_{ij} + a_{ji}\right) \, , & \gamma_i &= -\frac{1}{2}\varepsilon_{ijk} a_{jk} \\ \beta_{ij} &= \frac{1}{4}\left(\varepsilon_{ikl} b_{jkl} + \varepsilon_{jkl} b_{ikl}\right) \, , & \delta_i &= \frac{1}{4}\left(b_{jji} - b_{jij}\right) \end{aligned}$$

Mean field induction equation

Induction equation for $\overline{\boldsymbol{B}}$:

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left[\boldsymbol{\alpha}\overline{\boldsymbol{B}} + (\overline{\boldsymbol{v}} + \gamma) \times \overline{\boldsymbol{B}} - (\eta + \boldsymbol{\beta})\, \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \boldsymbol{\delta} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})\right]$$

Interpretation on first sight:

- $\boldsymbol{\alpha}$: new effect
- γ: acts like advection (turbulent advection effect)
- $\boldsymbol{\beta}$: acts like diffusion (turbulent diffusivity)
- $\boldsymbol{\delta}$: special anisotropy of diffusion tensor

Symmetry constraints

α, β, γ and δ depend on large-scale symmetries of the system defining the symmetry properties of the turbulence (e.g. rotation and stratification). Additional to that the expansion

$$\overline{\boldsymbol{\mathcal{E}}} = \boldsymbol{\alpha}\overline{\boldsymbol{B}} + \boldsymbol{\gamma} \times \overline{\boldsymbol{B}} - \boldsymbol{\beta}\boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \boldsymbol{\delta} \times \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} + \ldots$$

is a relation between polar and axial vectors:

- $\overline{\boldsymbol{\mathcal{E}}}$: polar vector, independent from handedness of coordinate system
- $\boldsymbol{B}$: axial vector, involves handedness of coordinate system in definition (curl operator, cross product)

Handedness of coordinate system pure convention (contains no physics), consistency requires:

- $\boldsymbol{\alpha}$, $\boldsymbol{\delta}$: pseudo tensor
- $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$: true tensors

Symmetry constraints

Turbulence with rotation and stratification

- true tensors: δ_{ij}, g_i, $g_i g_j$, $\Omega_i\Omega_j$, $\Omega_i\varepsilon_{ijk}$
- pseudo tensors: ε_{ijk}, Ω_i, $\Omega_i g_j$, $g_i\varepsilon_{ijk}$

Symmetry constraints allow only certain combinations:

$$
\begin{aligned}
\alpha_{ij} &= \alpha_0(\boldsymbol{g}\cdot\boldsymbol{\Omega})\delta_{ij} + \alpha_1\left(g_i\Omega_j + g_j\Omega_i\right)\,, \qquad & \gamma_i &= \gamma_0 g_i + \gamma_1\varepsilon_{ijk}g_j\Omega_k \\
\beta_{ij} &= \beta_0\,\delta_{ij} + \beta_1\,g_i g_j + \beta_2\,\Omega_i\Omega_j\,, & \delta_i &= \delta_0\Omega_i
\end{aligned}
$$

The scalars $\alpha_0\ldots\delta_0$ depend on quantities of the turbulence such as rms velocity and correlation times scale.

- isotropic turbulence: only $\boldsymbol{\beta}$
- $+$ stratification: $\boldsymbol{\beta}+\boldsymbol{\gamma}$
- $+$ rotation: $\boldsymbol{\beta}+\boldsymbol{\delta}$
- $+$ stratification $+$ rotation: $\boldsymbol{\alpha}$ can exist

Simplified expressions

Assuming $|\boldsymbol{B}'| \ll |\overline{\boldsymbol{B}}|$ in derivation + additional simplification for (quasi) isotropic, non-mirror symmetric, (weakly) inhomogeneous turbulence (see homework assignment):

$$\overline{v_i' v_j'} \sim \delta_{ij}, \ \alpha_{ij} = \alpha \delta_{ij}, \ \beta_{ij} = \eta_t \delta_{ij}$$

Leads to:

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left[\alpha \overline{\boldsymbol{B}} + (\overline{\boldsymbol{v}} + \gamma) \times \overline{\boldsymbol{B}} - (\eta + \eta_t) \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} \right]$$

with the scalar quantities

$$\alpha = -\frac{1}{3} \tau_c \overline{\boldsymbol{v}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}')}, \qquad \eta_t = \frac{1}{3} \tau_c \overline{\boldsymbol{v}'^2}$$

and vector

$$\gamma = -\frac{1}{6} \tau_c \boldsymbol{\nabla} \overline{\boldsymbol{v}'^2} = -\frac{1}{2} \boldsymbol{\nabla} \eta_t$$

Expressions are independent of η (in this approximation), indicating fast dynamo action!

Turbulent diffusivity - destruction of magnetic field

Turbulent diffusivity dominant dissipation process for large-scale field in case of large R_m:

$$\eta_t = \frac{1}{3}\tau_c \, \overline{\boldsymbol{v}'^2} \sim L \, v_{\rm rms} \sim R_m \eta \gg \eta$$

- Formally η_t comes from advection term (transport term, non-dissipative)
- Turbulent cascade transporting magnetic energy from the large scale L to the micro scale l_m (advection + reconnection)

$$\eta \boldsymbol{j}_m^2 \sim \eta_t \bar{\boldsymbol{j}}^2 \longrightarrow \frac{B_m}{l_m} \sim \sqrt{R_m}\frac{\overline{B}}{L}$$

Important: The large-scale determines the energy dissipation rate, l_m adjusts to allow for the dissipation on the microscale.

Present for isotropic homogeneous turbulence

Turbulent diamagnetism, turbulent pumping

Expulsion of flux from regions with larger turbulence intensity 'diamagnetism'

$$\boldsymbol{\gamma} = -\frac{1}{2}\boldsymbol{\nabla}\eta_t$$

Turbulent pumping (stratified convection):

$$\boldsymbol{\gamma} = -\frac{1}{6}\tau_c \boldsymbol{\nabla}\overline{\boldsymbol{v}'^2}$$

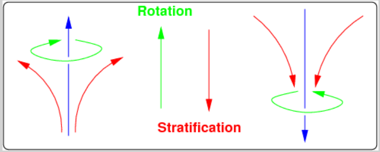
- Upflows expand, downflows converge
- Stronger velocity and smaller filling factor of downflows
- Mean induction effect of up- and downflow regions does not cancel
- Downward transport found in numerical simulations

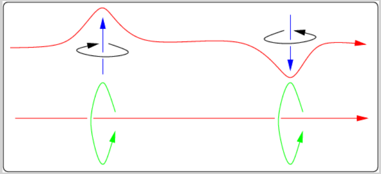
Requires inhomogeneity (stratification)

Kinematic α-effect

$$\alpha = -\frac{1}{3}\tau_c \overline{\boldsymbol{v}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}')} \qquad H_k = \overline{\boldsymbol{v}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}')} \text{ kinetic helicity}$$

Requires rotation + additional preferred direction (stratification)





Fast or slow dynamo?

Turbulent induction effects require reconnection to operate; however, the expressions

$$
\begin{aligned}
\alpha_{ij} &= \frac{1}{2}\tau_c \left(\varepsilon_{ikl} \overline{v_k{}' \frac{\partial v_l{}'}{\partial x_j}} + \varepsilon_{jkl} \overline{v_k{}' \frac{\partial v_l{}'}{\partial x_i}} \right) \\
\gamma_i &= -\frac{1}{2}\tau_c \frac{\partial}{\partial x_k} \overline{v_i' v_k'} \\
\beta_{ij} &= \frac{1}{2}\tau_c \left(\overline{\boldsymbol{v}'^2} \delta_{ij} - \overline{v_i{}' v_j{}'} \right)
\end{aligned}
$$

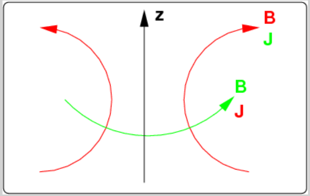
are independent of η (in this approximation), indicating fast dynamo action (no formal proof since we made strong assumptions!)

Meanfield energy equation

$$\frac{d}{dt}\int \frac{\overline{\boldsymbol{B}}^2}{2\mu_0}\,dV = -\mu_0 \int \eta \overline{\boldsymbol{j}}^2\,dV - \int \overline{\boldsymbol{v}} \cdot (\overline{\boldsymbol{j}} \times \overline{\boldsymbol{B}})\,dV + \int \overline{\boldsymbol{j}} \cdot \overline{\mathcal{E}}\,dV$$

- Energy conversion by α-effect $\sim \alpha \overline{\boldsymbol{j}} \cdot \overline{\boldsymbol{B}}$
- α-effect only pumps energy into meanfield if meanfield is helical (current helicity must have same sign as α)!
- Dynamo action does not necessarily require that $\overline{\boldsymbol{j}} \cdot \overline{\mathcal{E}}$ is an energy source. It can be sufficient if $\overline{\mathcal{E}}$ changes field topology to circumvent Cowling, if other energy sources like differential rotation are present (i.e. $\Omega \times \overline{\boldsymbol{j}}$ effect).

α^2-dynamo



Induction of field parallel to current (producing helical field!)

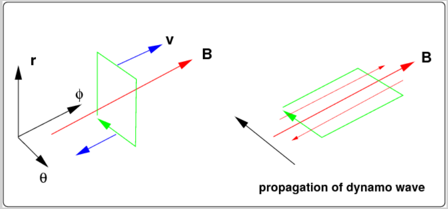
$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times (\alpha \overline{\boldsymbol{B}}) = \alpha \mu_0 \overline{\boldsymbol{j}}$$

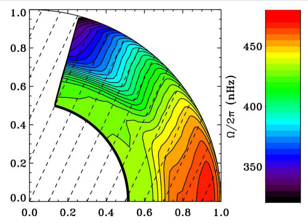
Dynamo cycle:

$$\boldsymbol{B}_t \xrightarrow{\alpha} \boldsymbol{B}_p \xrightarrow{\alpha} \boldsymbol{B}_t$$

- Poloidal and toroidal field of similar strength
- In general stationary solutions

$\alpha\Omega$-, $\alpha^2\Omega$-dynamo





Dynamo cycle:

$$\boldsymbol{B}_t \xrightarrow{\alpha} \boldsymbol{B}_p \xrightarrow{\Omega,\,\alpha} \boldsymbol{B}_t$$

- Toroidal field much stronger that poloidal field
- In general traveling (along lines of constant Ω) and periodic solutions

$\alpha\Omega$-dynamo

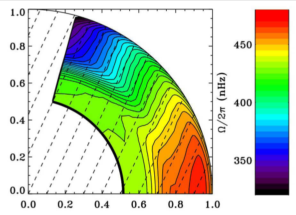
$$
\begin{aligned}
\frac{\partial B}{\partial t} &= r \sin\, \boldsymbol{B}_p \cdot \boldsymbol{\nabla}\Omega + \eta_t \left(\Delta - \frac{1}{(r \sin\theta)^2} \right) B \\
\frac{\partial A}{\partial t} &= \alpha B + \eta_t \left(\Delta - \frac{1}{(r \sin\theta)^2} \right) A
\end{aligned}
$$

- Cyclic behavior:

$$P \propto (\alpha |\boldsymbol{\nabla}\Omega|)^{-1/2}$$

- Propagation of magnetic field along contourlines of Ω “dynamo-wave”
- Direction of propagation “Parker-Yoshimura-Rule”:

$$\boldsymbol{s} = \alpha \boldsymbol{\nabla}\Omega \times \boldsymbol{e}_\phi$$



$\Omega \times J$ dynamo

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times [\delta \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})] \sim \boldsymbol{\nabla} \times (\Omega \times \bar{\boldsymbol{j}}) \sim \frac{\partial \bar{\boldsymbol{j}}}{\partial z}$$

- similar to α-effect, but additional z-derivative of current
- couples poloidal and toroidal field
- δ^2 dynamo is not possible:

$$\bar{\boldsymbol{j}} \cdot \overline{\mathcal{E}} = \bar{\boldsymbol{j}} \cdot (\boldsymbol{\delta} \times \bar{\boldsymbol{j}}) = 0$$

- δ-effect is controversial (not all approximations give a non-zero effect)
- in most situations α dominates

Dynamos and magnetic helicity

Magnetic helicity (integral measure of field topology):

$$H_m = \int \boldsymbol{A} \cdot \boldsymbol{B} \, dV$$

has following conservation law (no helicity fluxes across boundaries):

$$\frac{d}{dt} \int \boldsymbol{A} \cdot \boldsymbol{B} \, dV = -2\mu_0 \, \eta \int \boldsymbol{j} \cdot \boldsymbol{B} \, dV$$

Decomposition into contributions from small and large-scale magnetic field:

$$\begin{aligned} \frac{d}{dt} \int \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{B}} \, dV &= +2 \int \overline{\boldsymbol{\mathcal{E}}} \cdot \overline{\boldsymbol{B}} \, dV - 2\mu_0 \, \eta \int \overline{\boldsymbol{j}} \cdot \overline{\boldsymbol{B}} \, dV \\ \frac{d}{dt} \int \overline{\boldsymbol{A}' \cdot \boldsymbol{B}'} \, dV &= -2 \int \overline{\boldsymbol{\mathcal{E}}} \cdot \overline{\boldsymbol{B}} \, dV - 2\mu_0 \, \eta \int \overline{\boldsymbol{j}' \cdot \boldsymbol{B}'} \, dV \end{aligned}$$

Dynamos and magnetic helicity

Dynamos have helical fields:

- α-effect induces magnetic helicity of same sign on large-scale
- α-effect induces magnetic helicity of opposite sign on small scale

Asymptotic staturation

$$\begin{aligned}
\overline{\boldsymbol{j}' \cdot \boldsymbol{B}'} &= -\bar{\boldsymbol{j}} \cdot \overline{\boldsymbol{B}} \longrightarrow \frac{|\overline{B}|}{|B'|} \sim \sqrt{\frac{L}{l_c}} \\
\overline{\boldsymbol{j}' \cdot \boldsymbol{B}'} &= -\frac{\alpha \overline{\boldsymbol{B}}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \bar{\boldsymbol{j}} \cdot \overline{\boldsymbol{B}}
\end{aligned}$$

Non-kinematic effects

Proper way to treat them: 3D simulations

- Still very challenging, can't be done for the correct parameter regime
- Has been successful for geodynamo, but not for solar dynamo

Semi-analytical treatment of Lorentz-force feedback in mean field models:

- Macroscopic feedback: Change of the mean flow (differential rotation, meridional flow) through the mean Lorentz-force

$$\overline{\boldsymbol{f}} = \overline{\boldsymbol{j}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{j}' \times \boldsymbol{B}'}$$

- Mean field model including mean field representation of full MHD equations
- Microscopic feedback: Change of turbulent induction effects (e.g. α-quenching)

Microscopic feedback

Feedback of Lorentz force on small-scale motions:

- Intensity of turbulent motions significantly reduced if $\frac{1}{2\mu_0}B^2 > \frac{1}{2}\varrho v_{rms}^2$. Typical expression used

$$\alpha = \frac{\alpha_k}{1 + \frac{\overline{\boldsymbol{B}}^2}{B_{eq}^2}}$$

  with the equipartition field strength $B_{eq} = \sqrt{\mu_0 \varrho} v_{rms}$
- Similar quenching also expected for turbulent diffusivity
- Additional quenching of α due to topological constraints possible (helicity conservation)

Microscopic feedback

Symmetry of momentum and induction equation $\boldsymbol{v}' \leftrightarrow \boldsymbol{B}'/\sqrt{\mu_0 \varrho}$:

$$
\begin{aligned}
\frac{d\boldsymbol{v}'}{dt} &= \frac{1}{\mu_0 \varrho}(\overline{\boldsymbol{B}} \cdot \nabla)\boldsymbol{B}' + \dots \\
\frac{d\boldsymbol{B}'}{dt} &= (\overline{\boldsymbol{B}} \cdot \nabla)\boldsymbol{v}' + \dots \\
\overline{\boldsymbol{\mathcal{E}}} &= \overline{\boldsymbol{v}' \times \boldsymbol{B}'}
\end{aligned}
$$

Strongly motivates magnetic term for α-effect (Pouquet et al. 1976):

$$
\alpha = \frac{1}{3}\tau_c \left(\frac{1}{\varrho} \overline{\boldsymbol{j}' \cdot \boldsymbol{B}'} - \overline{\boldsymbol{\omega}' \cdot \boldsymbol{v}'} \right)
$$

- Kinetic α: $\overline{\boldsymbol{B}} + \boldsymbol{v}' \longrightarrow \boldsymbol{B}' \longrightarrow \overline{\boldsymbol{\mathcal{E}}}$
- Magnetic α: $\overline{\boldsymbol{B}} + \boldsymbol{B}' \longrightarrow \boldsymbol{v}' \longrightarrow \overline{\boldsymbol{\mathcal{E}}}$

Microscopic feedback

$$\alpha = \alpha_k + \frac{\tau_c}{3\varrho}\overline{\boldsymbol{j}' \cdot \boldsymbol{B}'}$$

With the asymptotic expression (steady state)

$$\overline{\boldsymbol{j}' \cdot \boldsymbol{B}'} = -\frac{\alpha \overline{\boldsymbol{B}}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta}\overline{\boldsymbol{j}} \cdot \overline{\boldsymbol{B}}$$

we get

$$\alpha = \frac{\alpha_\mathrm{k} + \frac{\eta_t^2}{\eta}\frac{\mu_0 \overline{\boldsymbol{j}} \cdot \overline{\boldsymbol{B}}}{B_\mathrm{eq}^2}}{1 + \frac{\eta_t}{\eta}\frac{\overline{\boldsymbol{B}}^2}{B_\mathrm{eq}^2}}$$

Microscopic feedback

Catastrophic α-quenching ($R_m \gg 1$!) in case of steady state and homogeneous $\overline{\boldsymbol{B}}$:

$$\alpha = \frac{\alpha_{\mathrm{k}}}{1 + R_m \frac{\overline{\boldsymbol{B}}^2}{B_{\mathrm{eq}}^2}}$$

If $\overline{\boldsymbol{j}} \cdot \overline{\boldsymbol{B}} \neq 0$ (dynamo generated field) and η_t unquenched:

$$\alpha \approx \eta_t\, \mu_0 \frac{\overline{\boldsymbol{j}} \cdot \overline{\boldsymbol{B}}}{\overline{\boldsymbol{B}}^2} \sim \frac{\eta_t}{L} \sim \frac{\eta_t}{l_c} \frac{l_c}{L} \sim \alpha_k \frac{l_c}{L}$$

- In general α-quenching dynamic process: linked to time evolution of helicity
- Boundary conditions matter: Loss of small-scale current helicity can alleviate catastrophic quenching
- Catastrophic α-quenching turns large-scale dynamo into slow dynamo

3D simulations

Why not just solving the full system to account for all non-linear effects?

- Most systems have $R_e \gg R_m \gg 1$, requiring high resolution
- Large-scale dynamos evolve on time scales $\tau_c \ll t \ll \tau_\eta$, requiring long runs compared to convective turn over
- 3D simulations successful for geodynamo
  - $R_m \sim 300$: all relevant magnetic scales resolvable
  - Incompressible system
- Solar dynamo: Ingredients can be simulated
  - Compressible system: density changes by 10^6 through convection zone
  - Boundary layer effects: Tachocline, difficult to simulate (strongly subadiabatic stratification, large time scales)
  - How much resolution required? (CZ about $\sim 10^9$ Mm^3, 1 Mm resolution $\sim 1000^3$ numerical problem)
  - Small-scale dynamos can be simulated (for $P_m \sim 1$)

Where did the “first” magnetic field come from?

Meanfield induction equation linear in $\overline{\boldsymbol{B}}$: possible solution.

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left[\alpha \overline{\boldsymbol{B}} + (\overline{\boldsymbol{v}} + \gamma) \times \overline{\boldsymbol{B}} - (\eta + \eta_t) \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}\right]$$

$\overline{\boldsymbol{B}} = 0$ is always a valid solution!
Generalized Ohm's law with electron pressure term:

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \frac{1}{\sigma}\boldsymbol{j} - \frac{1}{\varrho_e}\boldsymbol{\nabla} p_e \ .$$

leads to induction equation with inhomogeneous source term "Biermann Battery":

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B} - \eta \boldsymbol{\nabla} \times \boldsymbol{B}) + \frac{1}{\varrho_e^2}\boldsymbol{\nabla}\varrho_e \times \boldsymbol{\nabla} p_e \ .$$

Where did the "first" magnetic field come from?

Early universe:

- Ionization fronts from point sources (quasars) driven through an inhomogeneous medium: $1/\varrho_e^2 \nabla \varrho_e \times \nabla p_e$ can lead to about $10^{-23} G$
- Collapse of intergalactic medium to form galaxies leads to 10^{-20} G
- Galactic dynamo (growth rate $\sim 3 Gy^{-1}$) leads to 10^{-6} G after 10 Gy (today)

Source term is working all the time

- $\nabla \varrho_e \times \nabla p_e / \varrho^2$ at edge of solar granules induces field of about 10^{-6} G (Khomenko et al. 2017)

Next Lecture: Applications to Sun, Stars and planets

- Solar Dynamos
  - Large and small-scale flows in the solar convection zone
  - Overview of meanfield andf 3D dynamo models
  - Limitations of approaches
  - Small-scale dynamos
- Dynamos in solar-like stars
  - Effect of rotation and convection zone depth on dynamo properties
  - Evolution of stellar rotation and dynamos
- Geodynamo
  - What is similar, what is different compared to stellar dynamos?